

## NEW CONSTRUCTIONS OF TWISTOR LIFTS FOR HARMONIC MAPS

MARTIN SVENSSON AND JOHN C. WOOD

**ABSTRACT.** We show that given a harmonic map  $\varphi$  from a Riemann surface into a classical simply connected compact inner symmetric space, there is a  $J_2$ -holomorphic twistor lift of  $\varphi$  (or its negative) *if and only if* it is nilconformal. In the case of harmonic maps of finite uniton number, we give algebraic formulae in terms of holomorphic data which describes their extended solutions. In particular, this gives explicit formulae for the twistor lifts of all harmonic maps of finite uniton number from a surface to the above symmetric spaces.

## 1. INTRODUCTION

*Harmonic maps* are smooth maps between Riemannian manifolds which extremize the ‘Dirichlet’ energy integral (see, for example, [14]). Harmonic maps from surfaces into symmetric spaces are of particular interest to both geometers, as they include minimal surfaces, and to theoretical physicists, as they constitute the non-linear  $\sigma$ -model of particle physics. Twistor methods for finding such harmonic maps have been around for a long time; a general theory was given by F. E. Burstall and J. H. Rawnsley [7], see also [11]. The idea is to find a *twistor fibration* (for harmonic maps) — this is a fibration  $Z \rightarrow N$  from an almost complex manifold  $Z$ , called a *twistor space*, to a Riemannian manifold  $N$  with the property that holomorphic maps from (Riemann) surfaces to  $Z$  project to harmonic maps into  $N$ . For a symmetric space  $N$ , twistor spaces exist if  $N$  is *inner* [7]; then they are generalized flag manifolds equipped with a certain non-integrable complex structure  $J_2$ . All harmonic maps from the 2-sphere arise this way, i.e., have a *twistor lift* to a suitable flag manifold, see [7].

Burstall [5] showed that, given a harmonic map  $\varphi$  from a surface into a complex Grassmannian, there is a twistor lift of  $\varphi$  or its orthogonal complement  $\varphi^\perp$  if and only if  $\varphi$  is *nilconformal* in the sense that its derivative is nilpotent. We extend this result to other classical symmetric spaces as follows.

**Theorem 1.1.** *Let  $\varphi$  be a harmonic map from a surface into a classical compact simply connected inner symmetric space. Then there is a twistor lift of  $\varphi$  or  $\varphi^\perp$  if and only if  $\varphi$  is nilconformal.*

Any classical compact simply connected inner symmetric space is the product of irreducible ones; these are (i) the oriented real Grassmannians  $\tilde{G}_k(\mathbb{R}^n)$  with  $k(n-k)$  even, (ii) the complex and quaternionic Grassmannians, (iii) the space  $O(2m)/U(m)$  of orthogonal complex structures on  $\mathbb{R}^{2m}$  and (iv) the space  $Sp(m)/U(m)$  of ‘quaternionic’ complex structures on  $\mathbb{C}^{2m}$ . In cases (iii) and (iv),  $\varphi^\perp$  means the map  $-\varphi : p \mapsto -\varphi(p)$ .

Note that harmonic maps into oriented real Grassmannians with  $k(n-k)$  odd can be dealt with by embedding them in higher-dimensional Grassmannians (see Remark 6.8), and that harmonic maps into Grassmannians  $G_k(\mathbb{R}^n)$  of *unoriented* real subspaces are covered by those into oriented ones if a certain Steifel–Whitney class vanishes (Remark 6.15).

Nilconformal harmonic maps include all harmonic maps of finite uniton number (Example 4.2). They also include *strongly conformal* harmonic maps, in particular the superconformal harmonic

---

2000 *Mathematics Subject Classification.* 53C43, 58E20.

*Key words and phrases.* harmonic map, twistor, Grassmannian model, non-linear sigma model.

The first author was supported by the Danish Council for Independent Research under the project *Symmetry Techniques in Differential Geometry*. The second author thanks the Department of Mathematics and Computer Science of the University of Southern Denmark, Odense, for support and hospitality during part of the preparation of this work.

maps from the plane or a torus studied in [4, 3]; such maps from a torus are of finite type but not of finite unton number [26].

To establish our result, we introduce the idea of  $A_z^\varphi$ -filtrations, first of all for harmonic maps into complex Grassmannians, and show how these are related to twistor lifts, see Proposition 3.11. In fact, we can find such filtrations incorporating any given unton, giving us the existence of twistor lifts associated to that unton, see Theorem 5.8. Then we adapt our technique to the ‘real’ cases, showing in a constructive way how to build twistor lifts of harmonic maps from a surfaces to real Grassmannian, or to the space  $O(2m)/U(m)$ , see Propositions 6.14 and 6.21. Similar results hold for maps into quaternionic projective space or to the space  $Sp(m)/U(m)$ , see §7.1. Putting these results together gives our theorem.

In the case that  $\varphi$  has an *extended solution*  $\Phi$  (always true locally), we show how  $A_z^\varphi$ -filtrations are equivalent to certain other filtrations, called  $F$ -filtrations, of G. Segal’s Grassmannian model [32] of  $\Phi$ . We can then compute the twistor lift from the  $F$ -filtration and  $\Phi$ . We identify the  $F$ -filtration which gives Burstall’s twistor lift. When  $\varphi$  has finite unton number, we may choose  $\Phi$  to be polynomial; in that case, we have a natural  $F$ -filtration which leads to a new twistor lift, which we call the *canonical twistor lift*, see Theorem 4.8. Again, we can adapt these techniques to find twistor lifts of harmonic maps into the other classical simply connected inner symmetric space of type I, see Corollaries 6.10 and 6.19, and §7.1.

In the case of finite unton number we can do these constructions *explicitly*, as follows. In [20], simple formulae for the unitons of the factorization due to G. Segal were found, thus giving explicit algebraic formulae for all harmonic maps of finite unton number from a Riemann surface into the unitary group and complex Grassmannians, not involving any integration. Such formulae for K. Uhlenbeck’s factorization [34] — which is dual to that of Segal — appeared in [10]. In [33], it was shown how these formulae are extreme cases of a general method of finding explicit formulae for unton factorizations, and the method was adapted to construct harmonic maps into the orthogonal and symplectic groups, the real and quaternionic Grassmannians, and the spaces  $SO(2m)/U(m)$  and  $Sp(m)/U(m)$ , thus finding all harmonic maps into classical Lie groups and their inner symmetric spaces explicitly in terms of algebraic data.

In the present paper, we use those formulae to obtain explicit algebraic formulae for the  $J_2$ -holomorphic twistor lifts of arbitrary harmonic maps of finite unton number from a Riemann surface to a complex Grassmannian in terms of the freely chosen holomorphic data which give the unitons of the harmonic map. We then find the algebraic conditions on the holomorphic data which give the twistor lifts of harmonic maps into real and quaternionic Grassmannians, and into the spaces  $SO(2m)/U(m)$  and  $Sp(m)/U(m)$ . In particular, this gives explicit formulae for all harmonic maps from the two-sphere into the classical inner symmetric spaces and their twistor lifts.

## 2. PRELIMINARIES

**2.1. Harmonic maps into a Lie group.** Throughout the paper, all manifolds, bundles, and structures on them will be taken to be smooth, i.e.,  $C^\infty$ . By ‘Riemann surface’ we shall mean ‘connected 1-dimensional complex manifold’; we do not assume compactness. Harmonic maps from surfaces exhibit conformal invariance (see, for example, [37]) so that the concept of harmonic map from a Riemann surface is well defined. In the case of maps from a Riemann surface  $M$  to a Lie group  $G$ , we can formulate the harmonicity equations in the following way [34, 21].

For any smooth map  $\varphi : M \rightarrow G$ , set  $A^\varphi = \frac{1}{2}\varphi^{-1}d\varphi$ ; thus  $A^\varphi$  is a 1-form with values in the Lie algebra  $\mathfrak{g}$  of  $G$ ; note that it is half the pull-back of the Maurer–Cartan form of  $G$ .

Now, any compact Lie group can be embedded in the unitary group  $U(n)$ , so we first consider that group. The group  $U(n)$  acts on  $\mathbb{C}^n$  in the standard way. Let  $\underline{\mathbb{C}}^n$  denote the trivial complex bundle  $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$ , then  $D^\varphi = d + A^\varphi$  defines a unitary connection on  $\underline{\mathbb{C}}^n$ . We decompose  $A^\varphi$  and  $D^\varphi$  into types; for convenience we do this by taking a local complex coordinate  $z$  on an open set  $U$  of  $M$ . Explicitly, on writing  $d\varphi = \varphi_z dz + \varphi_{\bar{z}} d\bar{z}$ ,  $A = A_z^\varphi dz + A_{\bar{z}}^\varphi d\bar{z}$ ,  $D^\varphi = D_z^\varphi dz + D_{\bar{z}}^\varphi d\bar{z}$ ,

$\partial_z = \partial/\partial z$  and  $\partial_{\bar{z}} = \partial/\partial \bar{z}$ , we have

$$(2.1) \quad A_z^\varphi = \frac{1}{2}\varphi^{-1}\varphi_z, \quad A_{\bar{z}}^\varphi = \frac{1}{2}\varphi^{-1}\varphi_{\bar{z}}, \quad D_z^\varphi = \partial_z + A_z^\varphi, \quad D_{\bar{z}}^\varphi = \partial_{\bar{z}} + A_{\bar{z}}^\varphi.$$

By the *(Koszul–Malgrange) holomorphic structure* [24] induced by  $\varphi$  we mean the unique holomorphic structure on  $\underline{\mathbb{C}}^n$  with  $\bar{\partial}$ -operator given on each coordinate domain  $(U, z)$  by  $D_z^\varphi$ ; we denote the resulting holomorphic vector bundle by  $(\underline{\mathbb{C}}^n, D_z^\varphi)$ . If  $\varphi$  is constant, then  $D_z^\varphi = \partial_z$  giving  $\underline{\mathbb{C}}^n$  the *standard (product) holomorphic structure*. Uhlenbeck [34] showed that a smooth map  $\varphi : M \rightarrow G$  is harmonic if and only if, on each coordinate domain,  $A_z^\varphi$  is a holomorphic endomorphism of the holomorphic vector bundle  $(\underline{\mathbb{C}}^n, D_z^\varphi)$ . For later use, note that, if  $\varphi$  is replaced by  $g\varphi$  for some  $g \in \mathrm{U}(n)$ , then all the quantities in (2.1) are unchanged.

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . For any  $N \in \mathbb{N}$  and  $k \in \{0, 1, \dots, N\}$ , let  $G_k(\mathbb{C}^N)$  denote the Grassmannian of  $k$ -dimensional subspaces of  $\mathbb{C}^N$ ; it is convenient to write  $G_*(\mathbb{C}^N)$  for the disjoint union  $\bigcup_{k=0,1,\dots,N} G_k(\mathbb{C}^N)$ . We shall often identify, without comment, a smooth map  $\varphi : M \rightarrow G_k(\mathbb{C}^N)$  with the rank  $k$  subbundle of  $\underline{\mathbb{C}}^N = M \times \mathbb{C}^N$  whose fibre at  $p \in M$  is  $\varphi(p)$ ; we denote this subbundle also by  $\varphi$ , not underlining this as in, for example, [9, 19, 20].

For a subspace  $V$  of  $\mathbb{C}^n$  we denote by  $\pi_V$  (resp.  $\pi_V^\perp$ ) orthogonal projection from  $\mathbb{C}^n$  to  $V$  (resp. to its orthogonal complement  $V^\perp$ ); we use the same notation for orthogonal projection from  $\underline{\mathbb{C}}^n$  to a subbundle. Recall the Cartan embedding:

$$(2.2) \quad \iota : G_*(\mathbb{C}^n) \hookrightarrow \mathrm{U}(n), \quad \iota(V) = \pi_V - \pi_V^\perp;$$

this is totally geodesic, and isometric up to a constant factor. We shall identify  $V$  with its image  $\iota(V)$ ; since  $\iota(V^\perp) = -\iota(V)$ , this identifies  $V^\perp$  with  $-V$ .

**2.2. Harmonic maps into complex Grassmannians.** Harmonic maps into Grassmannians were studied by Burstall and the second author in [9] where the following definitions were made. Any subbundle  $\varphi$  of  $\underline{\mathbb{C}}^n$  inherits a metric by restriction, and a connection  $\nabla_\varphi$  by orthogonal projection:

$$(\nabla_\varphi)_Z(v) = \pi_\varphi(\partial_Z v) \quad (Z \in \Gamma(TM), v \in \Gamma(\varphi));$$

here  $\Gamma(\cdot)$  denotes the space of (smooth) sections of a vector bundle.

Let  $\varphi$  and  $\psi$  be two mutually orthogonal subbundles of  $\underline{\mathbb{C}}^n$ . Then, by the  $\partial'$  and  $\partial''$ -second fundamental forms of  $\varphi$  in  $\varphi \oplus \psi$  we mean the vector bundle morphisms  $A'_{\varphi,\psi}, A''_{\varphi,\psi} : \varphi \rightarrow \psi$  defined on each coordinate domain  $(U, z)$  by

$$(2.3) \quad A'_{\varphi,\psi}(v) = \pi_\psi(\partial_z v) \quad \text{and} \quad A''_{\varphi,\psi}(v) = \pi_\psi(\partial_{\bar{z}} v) \quad (v \in \Gamma(\varphi)).$$

The second fundamental forms  $A'_\varphi = A'_{\varphi,\varphi^\perp} : \varphi \rightarrow \varphi^\perp$  and  $A''_\varphi = A''_{\varphi,\varphi^\perp} : \varphi \rightarrow \varphi^\perp$  are particularly important as, on identifying  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  with its composition  $\iota \circ \varphi : M \rightarrow \mathrm{U}(n)$  with the Cartan embedding, it is easily seen that the fundamental endomorphism  $A_z^\varphi$  of (2.1) is minus the direct sum of  $A'_\varphi$  and  $A'_{\varphi^\perp}$ , similarly the connection  $D^\varphi$  of the last section is the direct sum of  $\nabla_\varphi$  and  $\nabla_{\varphi^\perp}$ . It follows that a smooth map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  is harmonic if and only if  $A'_\varphi$  is holomorphic, i.e.,  $A'_\varphi \circ \nabla_\varphi'' = \nabla_{\varphi^\perp}'' \circ A'_\varphi$ , where we write  $\nabla_\varphi'' = (\nabla_\varphi)_{\partial/\partial \bar{z}}$ ; this can be shown without reference to  $\mathrm{U}(n)$ , see [9, Lemma 1.3].

Now, for any holomorphic (or antiholomorphic) endomorphism  $E$ , at points where it does not have maximal rank, we shall ‘fill out zeros’ as in [9, Proposition 2.2] (cf. [33, §3.1]) to make its image and kernel into subbundles  $\mathrm{Im} E$  and  $\ker E$  of  $\underline{\mathbb{C}}^n$ . In particular, we obtain subbundles  $G'(\varphi) = \mathrm{Im} A'_\varphi$  and  $G''(\varphi) = \mathrm{Im} A''_\varphi$  called the  $\partial'$ - and  $\partial''$ -Gauss transforms or Gauss bundles of  $\varphi$ . Note that, if  $\varphi$  is harmonic, then so are its Gauss transforms. This can be seen by using diagrams as in [9, Proposition 2.3], or by noting that it is a special case of adding a uniton, cf. [37].

By iterating these constructions we obtain the  $i$ th  $\partial'$ -Gauss transform  $G^{(i)}(\varphi)$  defined by  $G^{(1)}(\varphi) = G'(\varphi)$ ,  $G^{(i)}(\varphi) = G'(G^{(i-1)}(\varphi))$ , and the  $i$ th  $\partial''$ -Gauss transform  $G^{(-i)}(\varphi)$  defined by  $G^{(-1)}(\varphi) = G''(\varphi)$ ,  $G^{(-i)}(\varphi) = G''(G^{(-i+1)}(\varphi))$ ; on setting  $G^{(0)}(\varphi) = \varphi$ , we obtain a sequence  $(G^{(i)}(\varphi))_{i \in \mathbb{Z}}$  (where  $\mathbb{Z}$  denotes the set of integers) of harmonic maps called [35] the *harmonic sequence of  $\varphi$* .

## 3. TWISTOR SPACES AND LIFTS

**3.1. Twistor spaces of complex Grassmannians.** Let  $N$  be a Riemannian manifold. By a *twistor fibration of  $N$  (for harmonic maps)* is meant [7] an almost complex manifold (called a *twistor space*)  $(Z, J)$  and a fibration  $\pi : Z \rightarrow N$  such that, for every holomorphic map from a Riemann surface  $\psi : M \rightarrow Z$ , the composition  $\varphi = \pi \circ \psi : M \rightarrow N$  is harmonic. (To deal with higher-dimensional domains, the definition is unchanged if we replace ‘Riemann surface’ by ‘cosymplectic manifold’, i.e., ‘almost Hermitian manifold with co-closed Kähler form’ [7, 31].) Then  $\varphi$  is called the *twistor projection of  $\psi$* , and  $\psi$  is called a *twistor lift of  $\varphi$* . For an even-dimensional Riemannian manifold  $N$ , the bundle  $Z \rightarrow N$  of almost Hermitian structures equipped with a suitable non-integrable almost complex structure  $J_2$  provides a twistor space, see [15] and [7, Chapter 2]. If  $N$  is orientable, we may consider the subbundle  $Z^+ \rightarrow N$  of *positive* almost Hermitian structures; however,  $Z$  and  $Z^+$  are usually too large to be useful and we look for subbundles of them.

For symmetric spaces, a general theory of such twistor fibrations is given in [7]. We shall now describe those twistor spaces for a complex Grassmannian; for the real and symplectic cases, see §6ff. For any complex vector spaces or vector bundles  $E, F$ ,  $\text{Hom}(E, F) = \text{Hom}_{\mathbb{C}}(E, F)$  will denote the vector space or bundle of (complex-)linear maps from  $E$  to  $F$ .

Let  $n, t, d_0, d_1, \dots, d_t$  be positive integers with  $\sum_{i=0}^t d_i = n$ . Let  $F = F_{d_0, \dots, d_t}$  be the (complex) flag manifold  $U(n)/U(d_0) \times \dots \times U(d_t)$ . The elements of  $F$  are  $(t+1)$ -tuples  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  of mutually orthogonal subspaces with  $\psi_0 \oplus \dots \oplus \psi_t = \mathbb{C}^n$ ; we call these subspaces the *legs (of  $\psi$ )*. There is a canonical embedding of  $F$  into the product  $G_{d_0}(\mathbb{C}^n) \times \dots \times G_{d_0+\dots+d_{t-1}}(\mathbb{C}^n)$  given by sending  $(\psi_0, \psi_1, \dots, \psi_t)$  to its *associated flag*  $(T_0, \dots, T_{t-1})$  where  $T_i = \sum_{j=0}^i \psi_j$ ; the restriction to  $F$  of the Kähler structure on this product is an (integrable) complex structure which we denote by  $J_1$ . Then the  $(1,0)$ -tangent space to  $F$  at  $(\psi_0, \psi_1, \dots, \psi_t)$  with respect to  $J_1$  is given by

$$(3.1) \quad T_{(1,0)}^{J_1} F = \sum_{0 \leq i < j \leq t} \text{Hom}(\psi_i, \psi_j).$$

Set  $k = \sum_{j=0}^{\lfloor t/2 \rfloor} d_{2j}$  and  $N = G_k(\mathbb{C}^n)$ . We define a mapping which gives the sum of the ‘even’ legs:

$$(3.2) \quad \pi_e : F_{d_0, \dots, d_t} \rightarrow G_k(\mathbb{C}^n), \quad \psi = (\psi_0, \psi_1, \dots, \psi_t) \mapsto \sum_{j=0}^{\lfloor t/2 \rfloor} \psi_{2j}.$$

The projection  $\pi_e$  is a Riemannian submersion with respect to the natural homogeneous metrics on  $F$  and  $G_*(\mathbb{C}^n)$  so that each tangent space decomposes into the orthogonal direct sum of the *vertical space*, made up of the tangents to the fibres and the *horizontal space*, its orthogonal complement. The  $(1,0)$ -horizontal and vertical spaces with respect to  $J_1$  are given by

$$\mathcal{H}_{(1,0)}^{J_1} = \sum_{\substack{i,j=0,\dots,t \\ i < j, j-i \text{ odd}}} \text{Hom}(\psi_i, \psi_j), \quad \mathcal{V}_{(1,0)}^{J_1} = \sum_{\substack{i,j=0,\dots,t \\ i < j, j-i \text{ even}}} \text{Hom}(\psi_i, \psi_j).$$

We define an almost complex structure  $J_2$  by changing the sign of  $J_1$  on the vertical space; thus the  $(1,0)$ -horizontal space is unchanged, but the  $(1,0)$  vertical space is different (unless  $t = 1$  when it is zero):

$$\mathcal{H}_{(1,0)}^{J_2} = \sum_{\substack{i,j=0,\dots,t \\ i < j, j-i \text{ odd}}} \text{Hom}(\psi_i, \psi_j), \quad \mathcal{V}_{(1,0)}^{J_2} = \sum_{\substack{i,j=0,\dots,t \\ j < i, j-i \text{ even}}} \text{Hom}(\psi_i, \psi_j).$$

This almost complex structure is never integrable except in the trivial case  $t = 1$ , see [8].

Now let  $M$  be a Riemann surface, which we shall always assume connected, but not necessarily compact, and let  $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M \rightarrow F$  be a smooth map; we shall call such a map a *moving flag*. From the above description we immediately obtain the following [5].

**Proposition 3.1.** *Let  $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M \rightarrow F$  be a smooth map. Then*

- (i)  *$\psi$  is  $J_1$ -holomorphic if and only if  $A'_{\psi_i, \psi_j} = 0$  whenever  $i - j$  is positive;*

(ii)  $\psi$  is  $J_2$ -holomorphic if and only if

$$(3.3) \quad A'_{\psi_i, \psi_j} = 0 \text{ when } i - j \text{ is positive and odd, or } j - i \text{ is positive and even.} \quad \square$$

**Remark 3.2.** (i) Using Proposition 3.1, it can be shown that  $\pi_e : (F, J_2) \rightarrow G_*(\mathbb{C}^n)$  is a twistor fibration, i.e., if  $\psi : M \rightarrow F$  is map from a Riemann surface which is holomorphic with respect to  $J_2$ , then the composition  $\varphi = \pi_e \circ \psi : M \rightarrow G_*(\mathbb{C}^n)$  is harmonic. This follows from the general theory of [7]; see [8] for a direct proof.

(ii) Let  $y \in G_k(\mathbb{C}^n)$ . For each  $\psi$  in the fibre  $(\pi_e)^{-1}(y)$ , the differential  $(d\pi_e)_\psi$  of the twistor projection at  $\psi$  restricts to an isometry of the horizontal space at  $\psi$  to  $T_y G_k(\mathbb{C}^n)$ . We can use this to transfer the almost complex structure  $J_1|_{\mathcal{H}} = J_2|_{\mathcal{H}}$  on the horizontal space to an almost Hermitian structure on  $T_y G_k(\mathbb{C}^n)$ . This defines an inclusion map  $i$  of  $F$  in the bundle  $Z \rightarrow N$  of almost Hermitian structures on  $G_k(\mathbb{C}^n)$ , see [29], showing how we may regard  $F$  as a subbundle of that bundle.

If now,  $\psi : M \rightarrow F$  is a  $J_1$  or  $J_2$ -holomorphic map, for each  $p \in M$ , the differential of  $\varphi$  at  $p$  intertwines the almost complex structure of  $M$  at  $p$  with the almost complex structure  $i \circ \psi(p)$  on  $T_{\varphi(p)} G_k(\mathbb{C}^n)$ ; thus the map  $\varphi$  is ‘rendered holomorphic’.

**3.2.  $J_2$ -holomorphic lifts of nilconformal maps from  $A_z^\varphi$ -filtrations.** We develop a general method for constructing  $J_2$ -holomorphic lifts from certain filtrations, which will culminate in Proposition 3.11. We first describe the filtrations we need; again,  $M$  will denote an arbitrary (connected) Riemann surface.

**Definition 3.3.** Let  $\varphi : M \rightarrow U(n)$  be a smooth map. Let  $(Z_i)$  be a finite sequence of subbundles of  $\underline{\mathbb{C}}^n$  which forms a *filtration* of  $\underline{\mathbb{C}}^n$ :

$$(3.4) \quad \underline{\mathbb{C}}^n = Z_0 \supset Z_1 \supset \cdots \supset Z_t \supset Z_{t+1} = \underline{0}.$$

We call  $(Z_i)$  an  $A_z^\varphi$ -filtration (of length  $t$ ) if, for each  $i = 0, 1, \dots, t$ ,

- (i)  $Z_i$  is a holomorphic subbundle, i.e.,  $\Gamma(Z_i)$  is closed under  $D_z^\varphi$ ;
- (ii)  $A_z^\varphi$  maps  $Z_i$  into the smaller subbundle  $Z_{i+1}$ .

Let  $\varphi : M \rightarrow U(n)$  be a smooth map. Say that  $\varphi$  is *nilconformal* if  $A_z^\varphi$  is nilpotent, i.e.,  $(A_z^\varphi)^r = 0$  for some non-negative integer  $r$ . Then  $A_z^\varphi$ -filtrations exist if and only if  $\varphi$  is nilconformal. Note that  $\varphi$  is nilconformal if and only if  $g\varphi$  is for any  $g \in U(n)$ . Burstall [5] calls a smooth map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  into a Grassmannian *nilconformal* if  $(A_z^\varphi)^r|_\varphi = 0$  for some  $r$ ; since this implies that  $(A_z^\varphi)^{r+1}|_{\varphi^\perp} = 0$ , nilconformality of  $\varphi$  in Burstall’s sense is equivalent to both nilconformality of  $\varphi^\perp$  in his sense and nilconformality of  $\varphi$  in our sense.

Any nilconformal map is weakly conformal; indeed, by nilpotency of  $A_z^\varphi$  we have  $\text{trace}(A_z^\varphi)^2 = 0$ , which is easily seen to be the condition for weak conformality (cf. [5]). Also, any harmonic map of finite uniton number is nilconformal, see Example 4.2 below.

For convenience, if  $E$  and  $F$  are subspaces of  $\mathbb{C}^N$ , or subbundles of  $\underline{\mathbb{C}}^N$  ( $N \in \mathbb{N}$ ), and  $F \subset E$ , we write  $E \ominus F$  to mean  $F^\perp \cap E$ .

Given a filtration  $(Z_i)$  of  $\underline{\mathbb{C}}^n$  of length  $t$ , we define its *legs*  $\psi_i$  by

$$(3.5) \quad \psi_i = Z_i \ominus Z_{i+1}, \quad \text{equivalently,} \quad Z_i = \sum_{j \geq i} \psi_j \quad (i = 0, 1, \dots, t+1).$$

Then the  $(t+1)$ -tuple  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  is an orthogonal decomposition of  $\underline{\mathbb{C}}^n$  into subbundles. If all subbundles are non-zero,  $\psi$  defines a smooth map from  $M$  to a flag manifold, i.e.,  $\psi$  is a moving flag as defined above; we continue to call it a moving flag even if some subbundles are zero. We extend  $\pi_e$  to the space of such moving flags so that we may continue to write  $\pi_e \circ \psi = \sum_j \psi_{2j}$ .

Now let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map to a Grassmannian. Say that a filtration (3.4) is *alternating* for  $\varphi$  if its legs  $\psi_i = Z_i \ominus Z_{i+1}$  satisfy

$$(3.6) \quad \psi_i \subset (-1)^i \varphi \quad \text{for } i = 0, 1, \dots, t.$$

This is equivalent to  $\varphi = \sum_j \psi_{2j}$ , i.e.,  $\pi_e \circ \psi = \varphi$ . We define *alternating for  $\varphi^\perp$*  similarly. By ‘alternating’ we shall mean ‘alternating for  $\varphi$  or  $\varphi^\perp$ ’.

**Lemma 3.4.** *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map. Then equation (3.5) defines a one-to-one correspondence between moving flags  $(\psi_0, \psi_1, \dots)$  which satisfy the  $J_2$ -holomorphicity condition (3.3) and  $A_z^\varphi$ -filtrations  $(Z_i)$  of  $\underline{\mathbb{C}}^n$  which are alternating for  $\varphi$ .*

*Proof.* Set  $U_i = \sum_{j \geq i} \psi_{2j} \subset \varphi$  and  $V_i = \sum_{j \geq i} \psi_{2j+1} \subset \varphi^\perp$ , so that  $Z_{2i} = U_i \oplus V_i$  and  $Z_{2i+1} = U_{i+1} \oplus V_i$ . It is easy to see that (3.3) is equivalent to

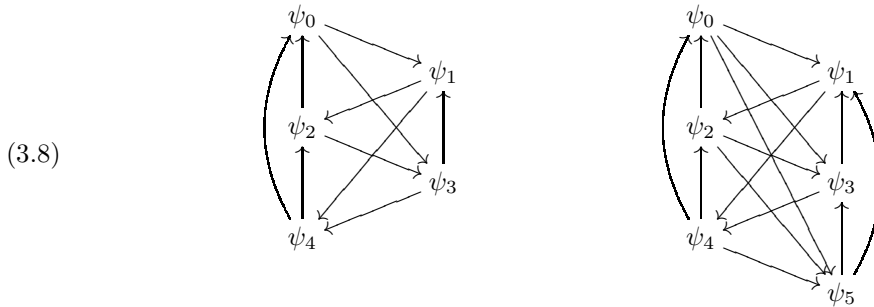
$$(3.7) \quad \begin{cases} \text{(i)} & U_i \text{ and } V_i \text{ are holomorphic subbundles of } \varphi \text{ and } \varphi^\perp, \text{ respectively, and} \\ \text{(ii)} & A'_\varphi(U_i) \subset V_i \text{ and } A'_{\varphi^\perp}(V_i) \subset U_{i+1}; \end{cases}$$

here, condition (i) means that  $\Gamma(U_i)$  is closed under  $\nabla''_\varphi$  and  $\Gamma(V_i)$  is closed under  $\nabla''_{\varphi^\perp}$ . The result follows by noting that conditions (3.7) (i) and (ii) are equivalent to conditions (i) and (ii) of Definition 3.3, respectively.  $\square$

Call a filtration (3.4) *strict* if all the inclusions  $Z_{i+1} \subset Z_i$  are strict; then we have the following result, illustrated by the diagrams (3.8); the length of the filtration  $(Z_i)$  is equal to 4 in the left-hand diagram and 5 in the right-hand one. In the diagrams, the vertices in the left (resp. right) columns represent the subbundles making up  $\varphi$  (resp.  $\varphi^\perp$ ). As in [9], the possible non-zero  $(\partial')$ -second fundamental forms  $A'_{\psi_i, \psi_j}$  are indicated by the arrows: more precisely, the absence of an arrow from  $\psi_i$  to  $\psi_j$  indicates that  $A'_{\psi_i, \psi_j} = 0$ . Note that the arrows indicating the only possible non-zero second fundamental forms  $A'_{\psi_i, \psi_j}$  are (i) vertically upwards when  $\psi_i$  and  $\psi_j$  are both in  $\varphi$ , or both in  $\varphi^\perp$ ; (ii) downwards from left to right when  $\psi_i \subset \varphi$ ,  $\psi_j \subset \varphi^\perp$ , in which case they are the components of  $A'_\varphi$ ; (iii) downwards from right to left when  $\psi_i \subset \varphi^\perp$ ,  $\psi_j \subset \varphi$ , in which case they are the components of  $A'_{\varphi^\perp}$ . From Lemma 3.4 we deduce the following.

**Proposition 3.5.** *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map. Then formulae (3.5) define a one-to-one correspondence between (i)  $J_2$ -holomorphic lifts  $(\psi_0, \psi_1, \dots) : M \rightarrow F$  of  $\varphi$  to a complex flag manifold (with all  $\psi_i$  non-zero) and (ii) strict  $A_z^\varphi$ -filtrations  $(Z_i)$  which are alternating for  $\varphi$ .*

*Further,  $F = F_{d_0, \dots, d_{t+1}}$  where  $(Z_i)$  has length  $t$ , and  $d_i = \text{rank } \psi_i$  ( $i = 0, 1, \dots, t+1$ ).*  $\square$



**Remark 3.6.** (i) The sets (i) and (ii) are empty unless  $\varphi$  is harmonic and nilconformal.

(ii) Let  $(Z_i)$  be a strict  $A_z^\varphi$ -sequence for a nilconformal harmonic map  $M \rightarrow U(n)$ . Then the legs  $\psi_i = Z_i \setminus Z_{i+1}$  define a moving flag and  $\tilde{\varphi} = \sum_j \psi_{2j} : M \rightarrow G_*(\mathbb{C}^n)$  defines a map into a Grassmannian. It would be interesting to study this map; however, examples show that it is not, in general, harmonic.

In practice, many  $A_z^\varphi$ -filtrations  $(Z_i)$  are not alternating but satisfy the following weaker condition. Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map. Say that the subbundle  $Z_i$  of  $\underline{\mathbb{C}}^n$  *splits* (for  $\varphi$ ) if it is a direct sum:

$$(3.9) \quad Z_i = U_i \oplus V_i \quad \text{where} \quad U_i \subset \varphi \quad \text{and} \quad V_i \subset \varphi^\perp.$$

Say that *the filtration  $(Z_i)$  splits* if each subbundle  $Z_i$  splits, equivalently, each  $\psi_i = Z_i \ominus Z_{i+1}$  splits:  $\psi_i = \psi_i \cap \varphi \oplus \psi_i \cap \varphi^\perp$ . It is convenient to write

$$(3.10) \quad \hat{\psi}_{2i} = U_i \ominus U_{i+1} = \psi_i \cap \varphi \quad \text{and} \quad \hat{\psi}_{2i+1} = V_i \ominus V_{i+1} = \psi_i \cap \varphi^\perp;$$

the resulting subbundles  $\hat{\psi}_i$  and possible second fundamental forms  $A'_{\hat{\psi}_i, \hat{\psi}_j}$  are shown for filtrations of lengths 1 and 2 in the diagrams (3.11). Note that some of the  $\hat{\psi}_i$  may be zero.

(3.11)

We now give four ways to convert an  $A_z^\varphi$ -filtration which splits into one which is alternating.

**Lemma 3.7.** *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map. Let  $(Z_i)$  be an  $A_z^\varphi$ -filtration which splits; denote its length by  $t$ . Set  $\psi_i = Z_i \ominus Z_{i+1}$  and define  $U_i, V_i$  and  $\hat{\psi}_i$  by (3.9) and (3.10).*

(i) *Set  $\tilde{Z}_{2j} = U_j \oplus V_j$  and  $\tilde{Z}_{2j+1} = U_{j+1} \oplus V_j$ . Then  $(\tilde{Z}_i)$  is an alternating  $A_z^\varphi$ -filtration of length  $2t + 1$  with legs*

$$(3.12) \quad (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_{2t}, \tilde{\psi}_{2t+1}) = (\hat{\psi}_0, \hat{\psi}_1, \dots, \hat{\psi}_{2t}, \hat{\psi}_{2t+1}).$$

(ii) *Reversing the roles of  $\varphi$  and  $\varphi^\perp$ , set  $\tilde{Z}_{2j} = U_j \oplus V_j$  and  $\tilde{Z}_{2j+1} = U_j \oplus V_{j+1}$ . Then  $(\tilde{Z}_i)$  is an alternating  $A_z^\varphi$ -filtration of length  $2t + 1$  with legs*

$$(3.13) \quad (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_{2t}, \tilde{\psi}_{2t+1}) = (\hat{\psi}_1, \hat{\psi}_0, \dots, \hat{\psi}_{2t+1}, \hat{\psi}_{2t}).$$

(iii) *Set  $\tilde{Z}_{2j} = U_{2j-1} \oplus V_{2j}$  where  $U_{-1} = \varphi$ , and  $\tilde{Z}_{2j+1} = U_{2j+1} \oplus V_{2j}$ . Then  $(\tilde{Z}_i)$  is an alternating  $A_z^\varphi$ -filtration of length  $t + 1$ . Its legs are given by*

$$(3.14) \quad (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_{t+1}) = \begin{cases} (\hat{\psi}_0, \hat{\psi}_1 + \hat{\psi}_3, \hat{\psi}_2 + \hat{\psi}_4, \hat{\psi}_5 + \hat{\psi}_7, \dots, \hat{\psi}_{2t-2} + \hat{\psi}_{2t}, \hat{\psi}_{2t+1}) & (t \text{ even}), \\ (\hat{\psi}_0, \hat{\psi}_1 + \hat{\psi}_3, \hat{\psi}_2 + \hat{\psi}_4, \hat{\psi}_5 + \hat{\psi}_7, \dots, \hat{\psi}_{2t-1} + \hat{\psi}_{2t+1}, \hat{\psi}_{2t}) & (t \text{ odd}). \end{cases}$$

(iv) *Reversing the roles of  $\varphi$  and  $\varphi^\perp$ , set*

$$\tilde{Z}_{2j} = U_{2j} \oplus V_{2j-1}, \quad \text{and} \quad \tilde{Z}_{2j+1} = U_{2j} \oplus V_{2j+1} \quad \text{where } V_{-1} = \varphi^\perp.$$

*Then  $(\tilde{Z}_i)$  is an alternating  $A_z^\varphi$ -filtration of length  $t + 1$ . Its legs are given by*

$$(3.15) \quad (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_{t+1}) = \begin{cases} (\hat{\psi}_1, \hat{\psi}_0 + \hat{\psi}_2, \hat{\psi}_3 + \hat{\psi}_5, \hat{\psi}_4 + \hat{\psi}_6, \dots, \hat{\psi}_{2t-1} + \hat{\psi}_{2t+1}, \hat{\psi}_{2t}) & (t \text{ even}), \\ (\hat{\psi}_1, \hat{\psi}_0 + \hat{\psi}_2, \hat{\psi}_3 + \hat{\psi}_5, \hat{\psi}_4 + \hat{\psi}_6, \dots, \hat{\psi}_{2t-2} + \hat{\psi}_{2t}, \hat{\psi}_{2t+1}) & (t \text{ odd}). \end{cases}$$

*Further, each of the above four moving flags (3.12)–(3.15) satisfies the  $J_2$ -holomorphicity condition (3.3).*

*Proof.* This follows from Lemma 3.4: that the four moving flags satisfy the hypotheses of that lemma is easily checked.  $\square$

Parts (iii) and (iv) are illustrated for  $t = 4$  by the left- and right-hand diagrams of (3.16), respectively. For clarity, the second fundamental forms  $A'_{\hat{\psi}_i, \hat{\psi}_j}$  between subbundles  $\hat{\psi}_i, \hat{\psi}_j$  which are both in  $\varphi$  or both in  $\varphi^\perp$  are not shown, and only the arrows of least gradient between subbundles  $\hat{\psi}_i, \hat{\psi}_j$ , with one in  $\varphi$  and the other in  $\varphi^\perp$ , are shown.

(3.16)

By Proposition 3.5, the four moving flags (3.12)–(3.15) define  $J_2$ -holomorphic lifts of  $\varphi$  or  $\varphi^\perp$  if all legs are non-zero. We now give a process for removing legs which are zero. Recall that, for a moving flag  $\psi = (\psi_0, \psi_1, \dots)$ , its twistor projection is given by  $\pi_e \circ \psi = \sum_j \psi_{2j}$ . The following is easily checked.

**Lemma 3.8.** *Let  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  be a moving flag with some legs equal to zero. Each of the following three operations gives a moving flag  $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_s)$  with  $s < t$ , fewer zero legs, and  $\pi_e \circ \tilde{\psi} = \pm \pi_e \circ \psi$ ; further, if  $\psi$  satisfies the  $J_2$ -holomorphicity condition (3.3), then so does  $\tilde{\psi}$ .*

Operation 1. *If the first leg  $\psi_0$  is zero, remove it and renumber:  $\tilde{\psi}_j = \psi_{j+1}$ , thus reducing the number of legs by one.*

Operation 2. *If the last leg  $\psi_t$  is zero, remove it, thus reducing the number of legs by one.*

Operation 3. *If any other leg  $\psi_i$  is zero, remove it and combine the legs on each side, giving a new lift with two fewer legs:*

$$\tilde{\psi}_j = \psi_j \quad (j < i - 1), \quad \tilde{\psi}_j = \psi_{i-1} + \psi_{i+1} \quad (j = i - 1), \quad \tilde{\psi}_j = \psi_{j+2} \quad (j > i - 1).$$

Note that, after Operation 1,  $\pi_e \circ \tilde{\psi} = -\pi_e \circ \psi$ ; after Operations 2 or 3,  $\pi_e \circ \tilde{\psi} = \pi_e \circ \psi$ .  $\square$

By iterating these operations, we obtain the following result.

**Proposition 3.9.** *Let  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  be a moving flag which satisfies (3.3). Set  $\varphi = \pi_e \circ \psi$ . Then we can remove and combine legs by the operations in Lemma 3.8 to obtain a moving flag  $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_s)$  with  $s \leq t$  and no  $\tilde{\psi}_i$  equal to zero, which satisfies (3.3), and has  $\pi_e \circ \tilde{\psi} = \pm \varphi$ ; thus  $\tilde{\psi} : M \rightarrow F$  is a  $J_2$ -holomorphic lift of  $\pm \varphi$ .  $\square$*

Even when the legs are non-zero, we can sometimes obtain lifts with fewer legs by a fourth operation as follows; for this, recall the definition (2.3) of the  $\partial'$ -second fundamental forms  $A'_{\psi_i, \psi_j}$ . Again, the proof is by direct checking.

**Lemma 3.10.** *Let  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  be a moving flag which satisfies (3.3), and set  $\varphi = \pi_e \circ \psi$ .*

Operation 4. *If  $A'_{\psi_i, \psi_{i+1}}$  is zero, replace  $\dots, \psi_{i-1}, \psi_i, \psi_{i+1}, \psi_{i+2}, \dots$  by  $\dots, \psi_{i-1} + \psi_{i+1}, \psi_i + \psi_{i+2}, \dots$*

*This gives a new moving flag  $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_s)$  with  $s \leq t$  and  $\pi_e \circ \tilde{\psi} = \varphi$  and satisfying (3.3). (Here, we set  $\psi_i$  equal to zero if  $i$  is outside the range  $0 \leq i \leq t$ .)  $\square$*

Note that Operation 4 reduces the number of legs by two unless  $i = 0$  or  $i + 1 = t$ , in which case it reduces the number of legs by one. By iterating this process, we can find a  $J_2$ -holomorphic map  $(\psi_0, \psi_1, \dots, \psi_t) : M \rightarrow F$  satisfying

$$(3.17) \quad A'_{\psi_i, \psi_{i+1}} \neq 0 \quad (i = 0, 1, \dots, t-1).$$

On putting the above results together, we obtain the main result of this section.



**Proposition 3.11.** *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a smooth map. Let  $(Z_i)$  be an  $A_z^\varphi$ -filtration which splits for  $\varphi$ . Set  $\psi_i = Z_i \ominus Z_{i+1}$  and write  $\psi_i = \hat{\psi}_{2i} \oplus \hat{\psi}_{2i+1}$  where  $\hat{\psi}_{2i} \subset \varphi$  and  $\hat{\psi}_{2i+1} \subset \varphi^\perp$ . Then there is a  $J_2$ -holomorphic twistor lift  $\psi = (\psi_0, \psi_1, \dots, \psi_t) : M \rightarrow F$  of  $\varphi$  or  $\varphi^\perp$  to a flag manifold  $F = F_{d_0, \dots, d_t}$  satisfying (3.17) with every leg  $\psi_i$  the sum of some of the  $\hat{\psi}_j$ .*

*Proof.* As in Lemma 3.7(i) the moving flag  $(\hat{\psi}_i)$  satisfies the  $J_2$ -holomorphicity condition. By carrying out Operations 1–4 as above, this can be modified to give a  $J_2$ -holomorphic lift with the stated properties.  $\square$

Now, for any nilconformal harmonic map, we can find  $A_z^\varphi$ -filtrations which split for  $\varphi$ , see the next examples; thus we obtain the following result, which also follows from the work of Burstall [5, Section 3].

**Corollary 3.12.** *A smooth map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  from a surface to a complex Grassmannian has a  $J_2$ -holomorphic twistor lift  $\psi : M \rightarrow F$  into a flag manifold if and only if it is harmonic and nilconformal.*

Using our methods, we can give a more detailed result, see Theorem 5.8.

We now explain how Burstall's construction fits into our theory. Recall that, when  $\varphi : M \rightarrow U(n)$  is a harmonic map,  $A_z^\varphi$  is a holomorphic endomorphism of  $(\mathbb{C}^n, D_z^\varphi)$ .

**Example 3.13.** Let  $\varphi : M \rightarrow U(n)$  be a nilconformal harmonic map so that  $(A_z^\varphi)^{t+1} = 0$  for some  $t \in \mathbb{N}$ . Set  $Z_0 = \mathbb{C}^n$  and  $Z_i = \text{Im}(A_z^\varphi|_{Z_{i-1}})$ . Then we obtain an  $A_z^\varphi$ -filtration:  $Z_i = \text{Im}(A_z^\varphi)^i$ , which we call the *filtration by  $A_z^\varphi$ -images*; note that  $Z_{t+1} = 0$  which implies that all inclusions  $Z_{i+1} \subset Z_i$  are strict. If  $\varphi$  is Grassmannian, this clearly splits, so we may apply part (iii) of Lemma 3.7 to obtain an alternating  $A_z^\varphi$ -filtration  $(\tilde{Z}_i)$  with legs (3.14). Alternatively, we may apply part (iv) to obtain an alternating  $A_z^\varphi$ -filtration  $(\tilde{Z}_i)$  with legs (3.15).

By strictness of the filtration, all the legs in (3.14) and (3.15) are non-zero, with the possible exception of the first ones. However, since  $Z_1 \neq Z_0$ , either (a)  $A'_{\varphi^\perp}$  is not surjective and  $\hat{\psi}_0$  is non-zero, in which case (3.14) gives a  $J_2$ -holomorphic lift of  $\varphi$ , or (b)  $A'_\varphi$  is not surjective and  $\hat{\psi}_1$  is non-zero, in which case (3.15) gives a  $J_2$ -holomorphic lift of  $\varphi^\perp$ ; for some  $\varphi$ , both (a) and (b) hold and we get both lifts. In case (a) we get  $\tilde{Z}_i = \tilde{U}_i + \tilde{V}_i$  with  $\tilde{U}_i = \text{Im}(A_z^\varphi)^{2i-1}|_{\varphi^\perp} = \text{Im}((A'_{\varphi^\perp} \circ A'_\varphi)^{i-1} \circ A'_{\varphi^\perp})$  and  $\tilde{V}_i = \text{Im}(A_z^\varphi)^{2i}|_{\varphi^\perp} = \text{Im}(A'_\varphi \circ A'_{\varphi^\perp})^i$  for  $i = 1, 2, \dots$ . The formulae for case (b) are obtained by interchanging  $\varphi$  and  $\varphi^\perp$ . This interprets a construction of Burstall [5, Section 3]; see also Example 4.10.

**Example 3.14.** Dually (cf. Example 6.3), set  $\hat{Z}_i = \ker(A_z^\varphi)^{t+1-i}$  so that  $\hat{Z}_{t+1} = 0$ . From  $(A_z^\varphi)^{t+1} = 0$ , we see that the filtration  $(Z_i)$  of the last example and the filtration  $(\hat{Z}_i)$  just defined satisfy  $Z_i \subset \hat{Z}_i$ ; however, these two filtrations are different, in general.

Note that  $A_z^\varphi|_\varphi \equiv A'_\varphi$  is zero if and only if  $\varphi$  is antiholomorphic, equivalently,  $\varphi^\perp$  is holomorphic. The first non-trivial case of a nilconformal map  $\varphi$  is when  $(A_z^\varphi)^2|_\varphi \equiv A'_{\varphi^\perp} \circ A'_\varphi$  is zero; we consider such maps now.

**Example 3.15.** A harmonic map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  is called *strongly conformal* if  $A'_{\varphi^\perp} \circ A'_\varphi = 0$ , equivalently,  $G'(\varphi)$  and  $G''(\varphi)$  are orthogonal; such maps are clearly nilconformal. If  $\varphi$  is a strongly conformal harmonic map which is neither holomorphic nor antiholomorphic, then  $G'(\varphi)$  and  $G''(\varphi)$  are non-zero orthogonal subbundles of  $\varphi^\perp$ , and we have twistor lifts as follows:

- (i) In Example 3.13, case (b) gives the  $J_2$ -holomorphic lift  $\psi = (G'(\varphi)^\perp \cap \varphi^\perp, \varphi, G'(\varphi))$  of  $\varphi^\perp$ .
- (ii) Similarly, we have a  $J_2$ -holomorphic lift of  $\varphi^\perp$  given by  $\psi = (G''(\varphi), \varphi, G''(\varphi)^\perp \cap \varphi^\perp)$ . This lift is dual to (i) in the sense that it is obtained from the lift in (i) by replacing the complex structure on the domain by its conjugate (cf. Example 6.3).

(iii) Examples (i) and (ii) are the extreme cases of the following. *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a strongly conformal harmonic map which is neither holomorphic nor antiholomorphic. Suppose that*

$$(3.18) \quad W \text{ is a holomorphic subbundle of } \varphi^\perp \text{ satisfying } \text{Im } A'_\varphi \subset W \subset \ker A'_{\varphi^\perp}.$$

Set  $V = W^\perp \cap \varphi^\perp$ . Then  $(V, \varphi, W)$  is a  $J_2$ -holomorphic lift of  $\varphi^\perp$ , and every  $J_2$ -holomorphic lift of  $\varphi^\perp$  with three legs is given this way. To see this, first note that the conditions (3.18) are equivalent to

$$(3.19) \quad V \text{ is an antiholomorphic subbundle of } \varphi^\perp \text{ satisfying } \operatorname{Im} A''_\varphi \subset V \subset \ker A''_{\varphi^\perp}.$$

Then note that these conditions are equivalent to saying that we have a diagram (3.20), where as usual, the arrows indicate the possible non-zero  $(\partial')$ -second fundamental forms.

$$(3.20) \quad \begin{array}{c} & V \\ & \uparrow \\ \varphi & \swarrow \searrow \\ & W \end{array}$$

Finally, from Proposition 3.1, we see that this diagram says precisely that  $\psi = (V, \varphi^\perp, W)$  is  $J_2$ -holomorphic.

(iv) On replacing  $\varphi$  by  $\varphi^\perp$ , we obtain the following from part (iii). *Let  $\varphi : M \rightarrow G_2(\mathbb{C}^n)$  be a harmonic map which is neither holomorphic nor antiholomorphic. Suppose that  $\varphi^\perp$  is strongly conformal. Then  $\varphi$  has a unique  $J_2$ -holomorphic lift  $\psi = (G''(\varphi^\perp), \varphi^\perp, G'(\varphi^\perp))$ .* See Corollary 7.4 for more information, and Example 5.11 for related examples.

**Example 3.16.** Consider the isometric minimal immersion of the torus  $\mathbb{C}/\langle 2\pi/\sqrt{3}, 2\pi i \rangle$  into  $\mathbb{CP}^2$  given by the harmonic map

$$(3.21) \quad \varphi(z) = [e^{z-\bar{z}}, e^{\zeta z - \bar{\zeta} \bar{z}}, e^{\zeta^2 z - \bar{\zeta}^2 \bar{z}}],$$

where  $\zeta = e^{2\pi i/3}$ . For  $i = 0, 1, 2$ , set  $\varphi_i$  equal to the  $i$ th  $\partial'$ -Gauss transform  $G^{(i)}(\varphi)$  (see §2.2). Then

$$\varphi_i(z) = [e^{z-\bar{z}}, \zeta^i e^{\zeta z - \bar{\zeta} \bar{z}}, \zeta^{2i} e^{\zeta^2 z - \bar{\zeta}^2 \bar{z}}];$$

in particular,  $G^{(3)}(\varphi) = \varphi$  showing that  $\varphi$  is *superconformal*. [4, 3].

It follows that  $\varphi$  is of *finite type*, see [4, Corollary 2.7]; such a map cannot be of finite uniton number by a result of R. Pacheco [26]. For this  $\varphi$ , the  $A_z^\varphi$ -filtrations  $(z_i)$  of Examples 3.13, 3.14 and 3.15 all coincide and are given by  $Z_0 = \underline{\mathbb{C}}^n$ ,  $Z_1 = \varphi \oplus \varphi_1$ ,  $Z_2 = \varphi_1$ ,  $Z_3 = \underline{0}$ ; the legs of this filtration define the  $J_2$ -holomorphic lift  $\psi = (\varphi_2, \varphi, \varphi_1) = (G''(\varphi), \varphi, G'(\varphi))$  of  $\varphi^\perp$ .

#### 4. TWISTOR LIFTS FROM EXTENDED SOLUTIONS

**4.1. Extended solutions.** Let  $G$  be a Lie group with identity element  $e$  and Lie algebra  $\mathfrak{g} \cong T_e G$ , and let  $\mathfrak{g}^\mathbb{C}$  denote the complexified Lie algebra  $\mathfrak{g} \otimes \mathbb{C}$ . Let  $\Omega G$  be the (based) loop group  $\Omega G = \{\gamma : S^1 \rightarrow G \text{ smooth} : \gamma(1) = e\}$ . Recall [34, 6] that a smooth map  $\Phi : M \rightarrow \Omega G$  is said to be an *extended solution* if, on any coordinate domain  $(U, z)$ , we have  $\Phi^{-1} \Phi_z = (1 - \lambda^{-1})A$ , for some map  $A : U \rightarrow \mathfrak{g}^\mathbb{C}$ .

For any map  $\Phi : M \rightarrow \Omega G$  and  $\lambda \in S^1$ , we define  $\Phi_\lambda : M \rightarrow G$  by  $\Phi_\lambda(p) = \Phi(p)(\lambda)$  ( $p \in M$ ). If  $\Phi : M \rightarrow \Omega G$  is an extended solution, the map  $\varphi = \Phi_{-1} : M \rightarrow G$  is harmonic and  $\varphi^{-1} \varphi_z = 2A$ ; on comparing with (2.1) we see that  $A = A_z^\varphi$ .

Conversely, given a harmonic map  $\varphi : M \rightarrow G$ , an extended solution  $\Phi : M \rightarrow \Omega G$  is said to be *associated* to  $\varphi$  if  $\Phi^{-1} \Phi_z = (1 - \lambda^{-1})A_z^\varphi$ , equivalently,  $\varphi = g\Phi_{-1}$  for some  $g \in G$ . Extended solutions associated to any given harmonic map always exist locally; they exist globally if the domain  $M$  is simply connected, for example, if  $M = S^2$ . Further, any two extended solutions  $\Phi, \tilde{\Phi}$  associated to the same harmonic map differ by a loop on their common domain  $D \subset M$ , i.e.,  $\tilde{\Phi} = \gamma\Phi$  for some  $\gamma \in \Omega G$ ; in particular, they are equal if they agree at some point of  $D$ .

Let  $\mathcal{H} = \mathcal{H}^{(n)}$  denote the Hilbert space  $L^2(S^1, \mathbb{C}^n)$ . By expanding into Fourier series, we have

$$\mathcal{H} = \text{linear closure of } \operatorname{span}\{\lambda^i e_j : i \in \mathbb{Z}, j = 1, \dots, n\},$$

where  $\{e_1, \dots, e_n\}$  is the standard basis for  $\mathbb{C}^n$ . Thus, elements of  $\mathcal{H}$  are of the form  $v = \sum_i \lambda^i v_i$  where each  $v_i \in \mathbb{C}^n$ ; we define projections  $P_i : \mathcal{H} \rightarrow \mathbb{C}^n$  by  $P_i(v) = v_i$  ( $i \in \mathbb{Z}$ ). If  $w = \sum_i \lambda^i w_i$  is another element of  $\mathcal{H}$ , its  $L^2$  inner product with  $v$  is given by  $\langle v, w \rangle = \sum_i v_i \bar{w}_i$ . The natural

action of  $U(n)$  on  $\mathbb{C}^n$  induces an action of  $\Omega U(n)$  on  $\mathcal{H}$  which is isometric with respect to this  $L^2$  inner product. We consider the closed subspace

$$\mathcal{H}_+ = \text{linear closure of } \text{span}\{\lambda^i e_j : i \in \mathbb{N}, j = 1, \dots, n\},$$

with orthogonal complement in  $\mathcal{H}$  given by

$$\mathcal{H}_+^\perp = \text{linear closure of } \text{span}\{\lambda^{-i} e_j : i = 1, 2, \dots, j = 1, \dots, n\}.$$

The action of  $\Omega U(n)$  on  $\mathcal{H}$  induces an action of  $\Omega U(n)$  on subspaces of  $\mathcal{H}$ ; denote by  $Gr = Gr^{(n)}$  the orbit of  $\mathcal{H}_+$  under that action. It is known from [27] that  $Gr$  consists of all the closed subspaces  $W \subset \mathcal{H}$  which enjoy the following properties:

- (i)  $W$  is closed under multiplication by  $\lambda$ , i.e.,  $\lambda W \subset W$ ;
- (ii) the orthogonal projection  $W \rightarrow \mathcal{H}_+$  is Fredholm;
- (iii) the orthogonal projection  $W \rightarrow \mathcal{H}_+^\perp$  is Hilbert–Schmidt;
- (iv) the images of the orthogonal projections  $W \rightarrow \mathcal{H}_+^\perp$  and  $W^\perp \rightarrow \mathcal{H}_+$  consist of smooth functions.

Furthermore, we have a bijective map

$$(4.1) \quad \Omega U(n) \ni \Phi \mapsto W = \Phi \mathcal{H}_+ \in Gr;$$

we call  $W$  the *Grassmannian model of  $\Phi$* . The map (4.1) restricts to a bijection from the *algebraic loop group*  $\Omega_{\text{alg}} U(n)$  consisting of those  $\gamma \in \Omega U(n)$  given by finite Laurent series:  $\gamma = \sum_{i=s}^r \lambda^i T_i$ , where  $r \geq s$  are integers and the  $T_i$  are  $n \times n$  complex matrices, to the set of  $\lambda$ -closed subspaces  $W$  of  $\mathcal{H}$  satisfying  $\lambda^r \mathcal{H}_+ \subset W \subset \lambda^s \mathcal{H}_+$ , for some integers  $r \geq s$ .

For  $r \in \mathbb{N}$ , let  $\Omega_r U(n)$  denote the set of polynomials  $\Phi = \sum_{k=0}^r \lambda^k T_k$  in  $\Omega_{\text{alg}} U(n)$  of degree at most  $r$ . Then (4.1) further restricts to a bijection from  $\Omega_r U(n)$  to the subset  $Gr_r \subset Gr$  of those  $\lambda$ -closed subspaces  $W$  of  $\mathcal{H}$  satisfying

$$(4.2) \quad \lambda^r \mathcal{H}_+ \subset W \subset \mathcal{H}_+.$$

Now let  $\Phi : M \rightarrow \Omega U(n)$  be a smooth map and set  $W = \Phi \mathcal{H}_+ : M \rightarrow Gr$ . We call  $W$  an *extended solution* if (i)  $W$  is holomorphic, i.e.  $\partial_{\bar{z}}(\Gamma(W)) \subset \Gamma(W)$ , and (ii)  $\Gamma(W)$  is closed under the operator  $F = \lambda \partial_z$ , i.e.,  $F(\Gamma(W)) \subset \Gamma(W)$ . Then [32],  $\Phi$  is an extended solution if and only if  $W$  is an extended solution.

A harmonic map  $\varphi : M \rightarrow U(n)$  is said to be of *finite uniton number* if it has a polynomial associated extended solution, then the minimum degree of such a polynomial  $\Phi$  is called the (*minimal*) *uniton number of  $\varphi$* ; note that we are not insisting that  $\varphi = \Phi_{-1}$  in this definition. All harmonic maps from  $S^2$  to  $U(n)$  are of finite uniton number [34]. Uhlenbeck further showed that, if  $\varphi : M \rightarrow U(n)$  has finite uniton number  $r$ , then  $r \leq n - 1$  and  $\varphi$  has a unique polynomial associated extended solution  $\Phi = \sum_{k=0}^r \lambda^k T_k$  of degree  $r$  with  $\text{Im } T_0$  full, i.e., not lying in any proper subspace. As in [23, 20, 33], we call this the *type one extended solution*; however, this concept does not seem to be useful for the real cases in §6ff.

**4.2. Finding  $J_2$ -holomorphic lifts from extended solutions.** When we have an extended solution  $\Phi$  for our harmonic map  $\varphi : M \rightarrow U(n)$ , we can get  $A_z^\varphi$ -filtrations, and thus twistor lifts, from suitable filtrations of the Grassmannian model of  $\Phi$ , as we now explain.

**Definition 4.1.** Let  $W = \Phi \mathcal{H}_+$  be an extended solution. Let  $(Y_i)$  be a sequence of  $\lambda$ -closed subbundles of  $\mathbb{C}^n$  with

$$(4.3) \quad W = Y_0 \supset Y_1 \supset \dots \supset Y_t \supset Y_{t+1} = \lambda W.$$

Call  $(Y_i)$  an *F-filtration (of  $W$  of length  $t$ )* if, for each  $i$ ,

- (i)  $Y_i$  is holomorphic, i.e.,  $\Gamma(Y_i)$  is closed under  $\partial_{\bar{z}}$ , and
- (ii)  $F = \lambda \partial_z$  maps sections of  $Y_i$  into sections of the smaller subbundle  $Y_{i+1}$ .

These conditions imply that each  $Y_i$  is an extended solution.

Now let  $W = \Phi\mathcal{H}_+$  be an extended solution and  $\varphi = \Phi_{-1} : M \rightarrow \mathbf{U}(n)$  the corresponding harmonic map. Consider the bundle morphism  $P_0 \circ \Phi^{-1} : W \rightarrow \mathbb{C}^n$  where  $P_0 : \mathcal{H} \rightarrow \mathbb{C}^n$  denotes projection onto the zeroth Fourier coefficient, as in §4.1. It follows from the extended solution equations (see [33, Proposition 3.9]) that the mapping  $P_0 \circ \Phi^{-1}$  intertwines the operators (i)  $\partial_{\bar{z}}$  with  $D_z^\varphi$  and (ii)  $F$  with  $-A_z^\varphi$ , i.e., we have the commutative diagrams (4.4), where the vertical arrows are surjective maps with kernel  $\Gamma(\lambda W)$ .

$$(4.4) \quad \begin{array}{ccc} \Gamma(W) & \xrightarrow{\partial_{\bar{z}}} & \Gamma(W) \\ P_0 \circ \Phi^{-1} \downarrow & & \downarrow P_0 \circ \Phi^{-1} \\ \Gamma(\mathbb{C}^n) & \xrightarrow{D_z^\varphi} & \Gamma(\mathbb{C}^n) \end{array} \quad \begin{array}{ccc} \Gamma(W) & \xrightarrow{F} & \Gamma(W) \\ P_0 \circ \Phi^{-1} \downarrow & & \downarrow P_0 \circ \Phi^{-1} \\ \Gamma(\mathbb{C}^n) & \xrightarrow{-A_z^\varphi} & \Gamma(\mathbb{C}^n) \end{array}$$

Now, given a filtration (4.3), we associate to it a filtration  $(Z_i)$  of  $\mathbb{C}^n$  by setting

$$(4.5) \quad Z_i = P_0 \circ \Phi^{-1} Y_i \quad (i = 0, 1, \dots, t+1).$$

Property (i) above says that *the bundle morphism  $P_0 \circ \Phi^{-1} : (W, \partial_{\bar{z}}) \rightarrow (\mathbb{C}^n, D_z^\varphi)$  is holomorphic*; this enables us to fill out zeros to make each  $Z_i$  a subbundle. Thus we have the commutative diagram (4.6).

$$(4.6) \quad \begin{array}{ccccccc} W = Y_0 & \supset & Y_1 & \supset & \dots & \supset & Y_t \supset Y_{t+1} = \lambda W \\ P_0 \circ \Phi^{-1} \downarrow & & \downarrow P_0 \circ \Phi^{-1} & & \downarrow P_0 \circ \Phi^{-1} & & \downarrow P_0 \circ \Phi^{-1} \\ \mathbb{C}^n = Z_0 & \supset & Z_1 & \supset & \dots & \supset & Z_t \supset Z_{t+1} = \mathbf{0} \end{array}$$

Here, each vertical map is a restriction of  $P_0 \circ \Phi^{-1} : W \rightarrow \mathbb{C}^n$  and is a surjective bundle morphism with kernel  $\lambda W$ ; further, each  $Y_i$  is the inverse image of  $Z_i$  under  $P_0 \circ \Phi^{-1}$ .

From Property (ii) above, we see that  *$(Y_i)$  is an  $F$ -filtration if and only if  $(Z_i)$  is an  $A_z^\varphi$ -filtration*. Thus, for an extended solution  $W = \Phi\mathcal{H}_+$  and corresponding harmonic map  $\varphi = \Phi_{-1}$ , *there is a one-to-one correspondence between  $F$ -filtrations of  $W$  and  $A_z^\varphi$ -filtrations of  $\mathbb{C}^n$ ; in particular,  $F$ -filtrations of  $W$  exist if and only if  $\varphi$  is nilconformal*.

We now give an important example of a  $F$ -filtration which will give us a canonical twistor lift for a harmonic map of finite uniton number. In the sequel, let  $r \in \mathbb{N}$ .

**Example 4.2.** Let  $\Phi$  be an extended solution. Suppose that this is polynomial of degree  $r$  so that  $W = \Phi\mathcal{H}_+$  satisfies (4.2). Set

$$(4.7) \quad Y_i = W \cap \lambda^i \mathcal{H}_+ + \lambda W \quad (i = 0, 1, \dots, r+1);$$

since  $\lambda^{r+1} \mathcal{H}_+ \subset \lambda W$ , we see that  $(Y_i)$  is an  $F$ -filtration of length  $r$ ; we shall call it the *canonical  $F$ -filtration for  $\Phi$* . Setting  $\varphi = \Phi_{-1}$ , we call the associated  $A_z^\varphi$ -filtration  $(Z_i)$  obtained from  $(Y_i)$  by (4.5) the *canonical  $A_z^\varphi$ -filtration for  $\Phi$* . See Theorem 4.8 for the resulting twistor lift; we calculate this in terms of unitons in Example 5.5.

This example shows that *if  $\varphi : M \rightarrow \mathbf{U}(n)$  is a harmonic map of finite uniton number  $r$ , then it is nilconformal with  $(A_z^\varphi)^{r+1} = 0$* .

To apply the above work to maps into a Grassmannian we now identify the appropriate class of extended solutions. Let  $\nu : \mathcal{H} \rightarrow \mathcal{H}$  be the involution  $\lambda \mapsto -\lambda$ . Then, as in [34, §8] and [32, §3],  $W = \Phi\mathcal{H}_+$  is closed under  $\nu$  if and only if  $\Phi$  is  $\nu$ -invariant in the sense that

$$(4.8) \quad \Phi_\lambda \Phi_{-1} = \Phi_{-\lambda} \quad (\lambda \in S^1);$$

this condition implies that the map  $\varphi = \Phi_{-1}$  satisfies  $\varphi^2 = I$ , which means that it has image in a complex Grassmannian  $G_*(\mathbb{C}^n)$ . Conversely, we have the following result.

**Lemma 4.3.** (i) *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a harmonic map from a Riemann surface which has an associated extended solution. Then it has a  $\nu$ -invariant extended solution  $\Psi$  with  $\Psi_{-1} = \varphi$ .*

(ii) *Suppose that  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  is a harmonic map of uniton number  $r$ . Then it has a  $\nu$ -invariant polynomial extended solution  $\Psi$  of degree  $r$  or  $r+1$  with  $\Psi_{-1} = \varphi$ .*

*Proof.* (i) Let  $\Phi$  be an associated extended solution of  $\varphi$ . Fix a base point  $z_0 \in M$ . By replacing  $\Phi$  by  $\Phi(z_0)^{-1}\Phi$ , we may assume that  $\Phi_\lambda(z_0) = I$  for all  $\lambda$ . Pick a homomorphism  $\gamma : S^1 \rightarrow U(n)$  with  $\gamma(-1) = \varphi(z_0)$ , for example  $\gamma(\lambda) = \pi_{\varphi(z_0)} + \lambda\pi_{\varphi(z_0)}^\perp$ , and define  $\Psi = \gamma\Phi$ . Then  $\Psi$  is an extended solution associated to  $\varphi$ , and since  $\Psi_{-1}(z_0) = \varphi(z_0)$ , we have  $\Psi_{-1} = \varphi$  everywhere. Now  $\Psi_{-\lambda}\Psi_{-1}^{-1}$  is also an extended solution associated to  $\varphi$  and  $\Psi_{-\lambda}(z_0)\Psi_{-1}^{-1}(z_0) = \gamma(-\lambda)\gamma(-1) = \gamma(\lambda) = \Psi_\lambda(z_0)$  for all  $\lambda$ , which implies that  $\Psi$  is  $\nu$ -invariant.

(ii) Let  $\Phi$  be the type one associated extended solution of  $\varphi$  (see §4.1), so that  $\varphi = Q\Phi_{-1}$  for some  $Q \in U(n)$ . Uhlenbeck shows [34, §15] (see also [20, Lemma 4.6]) that  $\varphi$  maps into a Grassmannian if and only if  $Q \in G_*(\mathbb{C}^n)$  and  $\Phi_\lambda = Q\Phi_{-\lambda}\Phi_{-1}^{-1}Q$ . Write  $Q = \pi_A - \pi_A^\perp$  where  $A$  is a subspace of  $\mathbb{C}^n$ ; note that if  $A = \mathbb{C}^n$  (resp.  $A = 0$ ) then  $Q = I$  (resp.  $Q = -I$ ). We see that  $\Psi = (\pi_A + \lambda\pi_A^\perp)\Phi$  is a  $\nu$ -invariant polynomial extended solution, of degree  $r$  or  $r + 1$ , with  $\Psi_{-1} = \varphi$ , as required.  $\square$

We remark that the uniton number of a harmonic map  $\varphi : M \rightarrow G_k(\mathbb{C}^n)$  of finite uniton number is at most  $\min(2k, 2n - 2k, n - 1)$  [12]; in fact [33, Corollary 5.7], we can find a polynomial extended solution  $\Phi$  of degree at most  $\min(2k, 2n - 2k, n - 1)$  with  $\Phi_{-1} = \pm\varphi$ . For some sharper estimates depending on the rank of  $A$ , see [19].

Next, we consider the effect on the filtrations of  $\nu$ -invariance. First note that, if  $W$  is closed under  $\nu$ , then  $W = W_+ \oplus W_-$  where  $W_\pm$  are the  $\pm 1$ -eigenspaces of  $\nu$ .

**Lemma 4.4.** *Let  $\Phi$  be a  $\nu$ -invariant extended solution and set  $W = \Phi\mathcal{H}_+$ . Let  $Y$  be a subbundle of  $W$  which contains  $\lambda W$ . Then  $Y$  is closed under  $\nu$  if and only if  $Z = P_0 \circ \Phi^{-1}Y$  splits, i.e., is the direct sum of subbundles  $Z_+$  and  $Z_-$  with  $Z_\pm \in \pm\varphi$ . In that case,  $Z_\pm = P_0 \circ \Phi^{-1}Y_\pm$ .*

*Proof.* The  $\nu$ -invariance condition (4.8) implies that  $\Phi_{-1} \circ P_0 \circ \Phi^{-1} = P_0 \circ \Phi^{-1} \circ \nu$ , i.e.,  $P_0 \circ \Phi^{-1}$  intertwines  $\nu$  with the involution  $\Phi_{-1} = \pi_\varphi - \pi_{\varphi^\perp}$  on  $\mathbb{C}^n$ , which establishes the lemma.  $\square$

Thus, in (4.6), each  $Z_i$  splits if and only if each  $Y_i$  is closed under  $\nu$ .

**Remark 4.5.** For an alternative point of view, given a filtration (4.3), set  $\hat{Y}_i = \pi(Y_i)$  ( $i = 0, 1, \dots, t + 1$ ), where  $\pi : W \rightarrow W/\lambda W$  denotes the natural projection. Since  $Y_i$  contains  $\lambda W$ , we have  $Y_i = \pi^{-1}(\hat{Y}_i)$ . The operator  $F$  descends to  $W/\lambda W$  and becomes tensorial; we call a filtration

$$W/\lambda W = \hat{Y}_0 \supset \hat{Y}_1 \supset \dots \supset \hat{Y}_t \supset \hat{Y}_{t+1} = \underline{0}$$

an  $F$ -filtration (of  $W/\lambda W$ ) if  $F$  maps  $\hat{Y}_i$  into  $\hat{Y}_{i+1}$ . Then  $Y_i$  is an  $F$ -filtration if and only if  $\hat{Y}_i$  is, and the isomorphism  $\Phi^{-1} : W/\lambda W \rightarrow \mathbb{C}^n$  gives a one-to-one correspondence between  $F$ -filtrations of  $W/\lambda W$  and  $A_\varphi^\nu$ -filtrations of  $\mathbb{C}^n$ . Since  $\Phi^{-1} \circ \pi = P_0 \circ \Phi^{-1}$ , we have  $Z_i = P_0 \circ \Phi^{-1}Y_i = \Phi^{-1}\hat{Y}_i$ . Lastly,  $\nu$  descends to  $W/\lambda W$ , and invariance of  $Y_i$  under  $\nu$  is equivalent to invariance of  $\hat{Y}_i$  under  $\nu$ . Hence, it would be natural to work in  $W/\lambda W$ ; however, for convenience, we continue to work in  $W$ .

For a harmonic map into a Grassmannian, we can choose an extended solution  $\Phi$  which is  $\nu$ -invariant; we now show that the canonical filtration for such a  $\Phi$  is alternating; see Example 5.5 for more information.

**Proposition 4.6.** *Let  $\Phi$  be a  $\nu$ -invariant polynomial extended solution of degree  $r$ . Set  $W = \Phi\mathcal{H}_+$  and  $\varphi = \Phi_{-1} : M \rightarrow G_*(\mathbb{C}^n)$ . Let  $(Z_i)$  be the canonical  $A_\varphi^\nu$ -filtration of Example 4.2, and set  $\psi_i = Z_i \ominus Z_{i+1}$  ( $i = 0, 1, \dots, r$ ). Then  $\psi_i \subset (-1)^i\varphi$  so that  $\varphi = \sum_j \psi_{2j}$ .*

*Proof.* Let  $x \in \psi_i = Z_i \ominus Z_{i+1}$ . Then  $x = P_0 \circ \Phi^{-1}(y)$  for some  $y \in W \cap \lambda^i \underline{\mathcal{H}}_+$ . Write  $y = y_+ + y_-$  where  $y_\pm \in W_\pm$ . Then  $x = x_+ + x_-$  where  $x_\pm = P_0 \circ \Phi^{-1}(y_\pm) \in \pm\varphi$ .

If  $i$  is even,  $y_- \in Y_{i+1}$ ; indeed, write  $y_- = y_1 + y_2$  where  $y_1 = P_i y_-$ , then applying  $\nu$  gives  $-y_- = y_1 + \nu y_2$ , so that  $y_1 = 0$ . Hence,  $x_- \in Z_{i+1}$ . Now  $x$  is orthogonal to  $Z_{i+1}$ ; it follows that  $0 = \langle x_+ + x_-, x_- \rangle = \langle x_-, x_- \rangle$  so that  $x_- = 0$  and  $x = x_+ \in \varphi$ . Similarly if  $i$  is odd,  $x_+ = 0$  and  $x = x_- \in -\varphi = \varphi^\perp$ .  $\square$

**Remark 4.7.** As in [33, §3.4], write  $A_i = \widehat{Y}_i \ominus \widehat{Y}_{i+1}$ ; the isometry  $\Phi : W/\lambda W \rightarrow \mathbb{C}^n$  maps  $A_i$  onto  $\psi_i$ . Proposition 4.6 is equivalent to saying that, for each  $i$ ,  $A_i$  is in the  $(-1)^i$ -eigenspace of  $\nu : W/\lambda W \rightarrow W/\lambda W$ .

By Lemma 3.4, the moving flag  $\psi = (\psi_0, \psi_1, \dots, \psi_r)$  defined in Proposition 4.6 satisfies the  $J_2$ -holomorphicity condition (3.3). To make  $\psi$  a twistor lift into a flag manifold, we need to ensure that each leg  $\psi_i$  is non-zero. As in [33, §3.4], say that a polynomial extended solution  $\Phi$  is *normalized* if each  $A_i$  is non-zero, equivalently, each  $\psi_i$  is non-zero. It is shown there that, if  $\Phi$  is not normalized, there is a polynomial loop  $\gamma$  such that  $\widetilde{\Phi} = \gamma^{-1}\Phi$  is a normalized polynomial extended solution of lesser degree; further, if  $\Phi$  is  $\nu$ -invariant, then we can choose the loop to be  $\nu$ -invariant, so that  $\widetilde{\Phi}$  is  $\nu$ -invariant and  $\widetilde{\Phi}_{-1} = \pm\Phi_{-1}$ . Recalling the definition of the flag manifold  $F_{d_0, d_1, \dots, d_r}$  from §3.1, we have the following result.

**Theorem 4.8.** *Let  $\Phi$  be a normalized  $\nu$ -invariant polynomial extended solution; denote its degree by  $r$  and set  $\varphi = \Phi_{-1}$ . Let the  $\psi_i$  be the legs of the canonical filtration as in Proposition 4.6 and set  $d_i = \text{rank } \psi_i$ . Then  $\psi = (\psi_0, \psi_1, \dots, \psi_r) : M \rightarrow F_{d_0, d_1, \dots, d_r}$  is a  $J_2$ -holomorphic lift of  $\varphi$ .*

We call  $\psi$  the *canonical (twistor) lift* of  $\varphi$  defined by  $\Phi$ , see Example 5.5 for a calculation of  $\psi$  in terms of unitons. Note that the canonical lift of  $\varphi$  depends on the choice of extended solution  $\Phi$ , however we have the following consequence.

**Corollary 4.9.** *Let  $\varphi : M \rightarrow G_k(\mathbb{C}^n)$  be a harmonic map of finite uniton number  $r$ . Then there is a  $J_2$ -holomorphic twistor lift  $\psi : M \rightarrow F$  of  $\varphi$  or  $\varphi^\perp$  into some flag manifold  $F = F_{d_0, d_1, \dots, d_t}$  with  $t \leq \min(r+1, 2k, 2n-2k, n-1)$ .*

*Proof.* By Lemma 4.3, there is a  $\nu$ -invariant polynomial extended solution  $\Phi$  of degree  $r$  or  $r+1$  with  $\Phi_{-1} = \varphi$ . If  $\Phi$  is not normalized, then by [33, Corollary 5.7], we can replace it by a normalized  $\nu$ -invariant polynomial extended solution  $\Psi$  with  $\Psi_{-1} = \pm\varphi$  of lesser degree, and that degree is at most  $\min(2k, 2n-2k, n-1)$ . This gives a twistor lift as in Theorem 4.8.  $\square$

Lastly, we shall find the  $F$ -sequence which leads to Burstall's twistor lift — note that this requires only that  $\varphi$  be nilconformal and not necessarily of finite uniton number.

**Example 4.10.** Let  $\varphi : M \rightarrow \text{U}(n)$  be nilconformal so that  $(A_z^\varphi)^{t+1} = 0$  for some  $t \in \mathbb{N}$ . Let  $\Phi$  be an associated extended solution of  $\varphi$  defined on an open subset of  $M$ . As shown in Lemma 4.3, we can take this to be  $\nu$ -invariant with  $\Phi_{-1} = \varphi$ . As usual, set  $W = \Phi\mathcal{H}_+$ . Set  $Y_0 = W$  and, for  $i = 1, 2, \dots$ , set  $Y_i = F(Y_{i-1}) + \lambda W$  (where we fill out zeros at points where the rank drops) so that  $Y_i = F^i(W) + \lambda W$ ; it follows that  $Y_{t+1} = \lambda W$ . The associated  $A_z^\varphi$ -filtration defined by (4.5) is the filtration  $Z_i = \text{Im}(A_z^\varphi)^i$  of Example 3.13 which, for a harmonic map into a Grassmannian, leads to Burstall's twistor lifts as explained in that example. Now, any two associated extended solutions  $\Phi$  and  $\widetilde{\Phi}$  differ by a loop on their common domain:  $\widetilde{\Phi} = \gamma\Phi$  for some  $\gamma \in \Omega\text{U}(n)$ . Since multiplication by a loop commutes with  $F$ , these give the same filtration  $(Z_i)$  on their common domain, so our construction is well defined on the whole of  $M$ .

When  $\varphi$  has finite uniton number, the twistor lifts arising from this construction are, in general, not the same as the canonical lift discussed above. See also Example 5.6.

## 5. TWISTOR LIFTS FROM UNITONS

**5.1. Unitons.** Let  $\varphi : M \rightarrow \text{U}(n)$  be a harmonic map. Then a subbundle  $\alpha$  of  $\mathbb{C}^n$  is said to be a *uniton* for  $\varphi$  if it is (i) holomorphic with respect to the Koszul–Malgrange holomorphic structure induced by  $\varphi$ , i.e.,  $D_z^\varphi(\sigma) \in \Gamma(\alpha)$  for all  $\sigma \in \Gamma(\alpha)$ ; and (ii) closed under the endomorphism  $A_z^\varphi$ , i.e.,  $A_z^\varphi(\sigma) \in \alpha$  for all  $\sigma \in \alpha$ . Uhlenbeck showed [34] that, if a subbundle  $\alpha \subset \mathbb{C}^n$  is a uniton for a harmonic map  $\varphi$ , then (i)  $\widetilde{\varphi} = \varphi(\pi_\alpha - \pi_\alpha^\perp)$  is harmonic, (ii)  $\alpha^\perp$  is a uniton for  $\widetilde{\varphi}$ , and (iii)  $\varphi = -\widetilde{\varphi}(\pi_\alpha^\perp - \pi_\alpha)$ .

**Example 5.1.** Any holomorphic subbundle of  $(\mathbb{C}^n, D_z^\varphi)$  contained in  $\ker A_z^\varphi$  is a uniton for  $\varphi$ ; we call such a uniton *basic*. Any holomorphic subbundle of  $(\mathbb{C}^n, D_z^\varphi)$  containing  $\text{Im } A_z^\varphi$  is also a uniton, we call such a uniton *antibasic*. Note that, if  $\alpha$  is basic (resp. antibasic) for  $\varphi$ , then  $\alpha^\perp$  is antibasic (resp. basic) for  $\widetilde{\varphi} = \varphi(\pi_\alpha - \pi_\alpha^\perp)$ .

Now suppose that  $\Phi$  is an extended solution associated to  $\varphi$  and  $\alpha$  is a subbundle of  $\underline{\mathbb{C}}^n$ , then Uhlenbeck showed that  $\alpha$  is a *uniton* for  $\varphi$  if and only if  $\tilde{\Phi} = \Phi(\pi_\alpha + \lambda\pi_\alpha^\perp)$  is an extended solution (associated to  $\tilde{\varphi} = \varphi(\pi_\alpha - \pi_\alpha^\perp)$ ); therefore, we shall also say that  $\alpha$  is a *uniton* for  $\Phi$ .

As before, let  $r \in \mathbb{N}$ . Let  $\Phi$  be a polynomial extended solution of degree at most  $r$ ; set  $W = \Phi\mathcal{H}_+$ . By a *partial uniton factorization* of  $\Phi$  we mean a product

$$(5.1) \quad \Phi = \Phi_0(\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp) \cdots (\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^\perp)$$

where (i)  $\Phi_0$  is an extended solution, and (ii) writing

$$(5.2) \quad \Phi_i = \Phi_0(\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp) \cdots (\pi_{\alpha_i} + \lambda\pi_{\alpha_i}^\perp) \quad (i = 1, 2, \dots, r),$$

each  $\Phi_i$  is an extended solution, and  $\alpha_i$  is a uniton for  $\Phi_{i-1}$  equivalently  $\alpha_i^\perp$  is a uniton for  $\Phi_i$ .

Note that each  $\varphi_i = (\Phi_i)_{-1}$  is harmonic and condition (ii) can be phrased as follows:  $\alpha_i$  is a uniton for  $\varphi_{i-1}$ , equivalently,  $\alpha_i^\perp$  is a uniton for  $\varphi_i$ .

Work of Segal [32] implies that setting  $W_i = \Phi_i\mathcal{H}_+$  defines an equivalence between partial uniton factorizations (5.1) and filtrations

$$(5.3) \quad W = W_r \subset W_{r-1} \subset \cdots \subset W_0 \subset \underline{\mathcal{H}}_+$$

by extended solutions satisfying

$$(5.4) \quad \lambda W_{i-1} \subset W_i \subset W_{i-1} \quad (i = 1, 2, \dots, r).$$

If  $\Phi_0 = I$ , equivalently,  $W_0 = \underline{\mathcal{H}}_+$ , then (5.1) is a *uniton factorization* in the sense of Uhlenbeck [34]. The argument in [33, §2] extends immediately to partial factorizations to show that the unitons in (5.1) are given by  $\alpha_i = P_0\Phi_{i-1}^{-1}W_i$ .

**5.2.  $J_2$ -holomorphic lifts from unitons.** Again, let  $\Phi$  be a polynomial extended solution of degree at most  $r$  and set  $W = \Phi\mathcal{H}_+$ . Let  $\varphi = \Phi_{-1}$  be the corresponding harmonic map. Given a (partial) uniton factorization (5.1), let  $W_i = \Phi_i\mathcal{H}_+$  be the associated filtration and set

$$(5.5) \quad Y_i = \lambda^i W_{r-i} + \lambda W \quad (i = 0, 1, \dots, r), \quad Y_{r+1} = \lambda W.$$

Then we have a filtration (4.3) of length  $r$ . We ask under what conditions it forms an  $F$ -filtration.

**Proposition 5.2.** *Let  $\Phi$  be a polynomial extended solution of degree at most  $r$  and (5.1) a partial uniton factorization.*

(i) *Suppose that*

$$(5.6) \quad F(\Gamma(W_i)) \subset \Gamma(\lambda W_{i-1}) \quad (i = 1, 2, \dots, r);$$

*then the filtration (5.5) is an  $F$ -filtration.*

(ii) *The condition (5.6) holds if and only if, for each  $i \in \{1, 2, \dots, r\}$ ,  $\alpha_i$  is a basic uniton for  $\Phi_{i-1}$ , equivalently,  $\alpha_i^\perp$  is an antibasic uniton for  $\Phi_i$ .*

(iii) *Let  $\alpha_1, \dots, \alpha_r$  be the unitons in (5.1). The composition  $\pi_{\alpha_r}^\perp \circ \cdots \circ \pi_{\alpha_{r-i+1}}^\perp$  is a holomorphic endomorphism from  $(\underline{\mathbb{C}}^n, D_z^{\varphi_{r-i}})$  to  $(\underline{\mathbb{C}}^n, D_z^\varphi)$ , and the  $A_z^\varphi$ -filtration  $(Z_i)$  associated to  $(Y_i)$  via (4.5) is given by*

$$(5.7) \quad Z_i = \text{Im}(\pi_{\alpha_r}^\perp \circ \cdots \circ \pi_{\alpha_{r-i+1}}^\perp), \quad \text{equivalently,} \quad Z_i^\perp = \ker(\pi_{\alpha_{r-i+1}}^\perp \circ \cdots \circ \pi_{\alpha_r}^\perp)$$

*( $i = 1, 2, \dots, r$ ). In particular,  $Z_1^\perp = \alpha_r$  and  $Z_2^\perp = \alpha_r + \alpha_r^\perp \cap \alpha_{r-1}$ .*

*Proof.* (i) From (5.6), we deduce that, for  $i \in \{0, 1, \dots, r-1\}$ ,

$$F(\Gamma(Y_i)) = F(\Gamma(\lambda^i W_{r-i} + \lambda W)) \subset \Gamma(\lambda^{i+1} W_{r-i-1} + \lambda W) = \Gamma(Y_{i+1}).$$

Further,  $F(\Gamma(Y_r)) \subset \Gamma(\lambda^{r+1} \underline{\mathcal{H}}_+) \subset \Gamma(\lambda W)$ .

(ii) This follows from the correspondence of the operators  $F$  and  $-A_z^\varphi$ , explained in §4.1, more precisely it is [33, Lemma 3.11] applied to  $\varphi = \varphi_i$ .

(iii) Using (5.5) and noting that  $P_0 \circ \Phi^{-1}(\lambda W) = 0$  and filling out zeros, we have

$$\begin{aligned} Z_i &= P_0 \circ \Phi^{-1}(Y_i) = P_0(\pi_{\alpha_r} + \lambda^{-1}\pi_{\alpha_r}^\perp) \cdots (\pi_{\alpha_{r-i+1}} + \lambda^{-1}\pi_{\alpha_{r-i+1}}^\perp) \Phi_{r-i}^{-1}(\lambda^i W_{r-i}) \\ &= P_0 \lambda^i (\pi_{\alpha_r} + \lambda^{-1}\pi_{\alpha_r}^\perp) \cdots (\pi_{\alpha_{r-i+1}} + \lambda^{-1}\pi_{\alpha_{r-i+1}}^\perp) \underline{\mathcal{H}}_+ \\ &= \pi_{\alpha_r}^\perp \circ \cdots \circ \pi_{\alpha_{r-i+1}}^\perp (\mathbb{C}^n). \end{aligned}$$

This gives the first formula of (5.7); taking the adjoint gives the second one.  $\square$

**Corollary 5.3.** *Let  $W = \Phi \mathcal{H}_+$  be a polynomial extended solution of degree  $r$  and let (5.1) be a partial uniton factorization of  $\Phi$  with corresponding filtration (5.3) which satisfies (5.6). As usual, define  $Y_i$  by (5.5), set  $Z_i = P_0 \circ \Phi^{-1} Y_i$  and write  $\psi_i = Z_i \ominus Z_{i+1}$  ( $i = 0, 1, \dots, r$ ).*

- (i) *The  $Z_i$  are given in terms of the unitons in (5.1) by (5.7); in particular,  $\psi_0 = \alpha_r$  and  $\psi_1 = \alpha_r^\perp \cap \alpha_{r-1}$ .*
- (ii) *Suppose that  $\Phi$  is  $\nu$ -invariant and that  $(Z_i)$  is a strict alternating filtration. Set  $d_i = \text{rank } \psi_i$ . Then  $\psi = (\psi_0, \psi_1, \dots, \psi_r) : M \rightarrow F_{d_0, d_1, \dots, d_r}$  is a  $J_2$ -holomorphic twistor lift of  $\varphi = \Phi_{-1} : M \rightarrow G_*(\mathbb{C}^n)$ .*

If (5.6) does not hold, then we cannot expect the filtration (5.5) to be an  $F$ -filtration, as shown by the following example.

**Example 5.4.** Let  $\Phi$  be a polynomial extended solution of degree  $r$  and set  $W = \Phi \mathcal{H}_+$ . Set  $W_i = W + \lambda^i \mathcal{H}_+$ . Then  $(W_i)$  defines a filtration (5.3) satisfying (5.4), and so a uniton factorization (5.1); we call this the *Segal factorization of  $\Phi$  (or  $W$ )*, as it appears in [32]. Defining  $\Phi_i$  by (5.2), we have  $W_i = \Phi_i \mathcal{H}_+$ . Each  $\alpha_i$  appearing in (5.1) is a uniton for  $\Phi_{i-1}$ ; we call the  $\alpha_i$  the *Segal unitons of  $\Phi$* .

The resulting filtration (5.5) is not, in general, an  $F$ -filtration; indeed, the Segal unitons are not basic in general, in fact, each  $\alpha_i$  is antibasic for  $\Phi_{i-1}$ .

However, there are lots of uniton factorizations with basic unitons to which we can apply Proposition 5.2(i) to give  $F$ -filtrations. We start with the factorization which gives the canonical twistor lift; then we identify the one which leads to Burstall's twistor lift.

**Example 5.5.** Let  $\Phi$  be a polynomial extended solution of degree at most  $r$  and set  $W = \Phi \mathcal{H}_+$ . Set  $W_i = \lambda^{r-i} W \cap \underline{\mathcal{H}}_+$ . Again,  $(W_i)$  defines a filtration (5.3) satisfying (5.4), and so a uniton factorization (5.1); we call this the *Uhlenbeck factorization of  $\Phi$  (or  $W$ )*, as it appears in [34]. Again, defining  $\Phi_i$  by (5.2), we have  $W_i = \Phi_i \mathcal{H}_+$  and each  $\alpha_i$  appearing in (5.1) is a uniton for  $\Phi_{i-1}$ ; we call the  $\alpha_i$  the *Uhlenbeck unitons of  $\Phi$* .

This time, the filtration  $(W_i)$  clearly satisfies (5.6), equivalently, each Uhlenbeck uniton  $\alpha_i$  is basic for  $\Phi_{i-1}$ . It is quickly checked that the  $F$ -filtration  $(Y_i)$  associated to  $(W_i)$  by (5.5) is the *canonical  $F$ -filtration* (4.7) which leads to the canonical twistor lift of Theorem 4.8.

**Example 5.6.** Let  $\Phi$  be a polynomial extended solution of degree  $r$ , and set  $W = \Phi \mathcal{H}_+$ . For any  $i \in \mathbb{N}$ , let  $W_{(i)}$  denote the  $i$ th osculating space spanned by derivatives of local holomorphic sections of  $W$  up to order  $i$ . Setting  $W_i = W_{(r-i)}$  defines a partial uniton factorization:

$$(5.8) \quad W = W_{(0)} \subset W_{(1)} \subset \cdots \subset W_{(r)} \subset \mathcal{H}_+$$

which satisfies (5.6). The proof in [33, Example 4.7] extends immediately to show that the unitons  $\alpha_i$  in (5.1) are given by  $\alpha_i^\perp = \text{Im } A_z^{\varphi_i}$ ; by definition,  $\alpha_i^\perp$  is antibasic for  $\varphi_i$  so  $\alpha_i$  is basic for  $\varphi_{i-1}$ . Defining  $Y_i$  by (5.5) gives the  $F$ -filtration  $Y_i = F(Y_{i-1}) + \lambda W = F^i(W) + \lambda W$  of Example 4.10; note that this formula automatically gives  $Y_{r+1} = \lambda W$  since  $F^{r+1}(W) \subset \lambda^{r+1} \underline{\mathcal{H}}_+ \subset \lambda W$ . The associated filtration  $(Z_i)$  defined by (4.5) is the filtration  $Z_i = \text{Im}(A_z^{\varphi})^i$  of Example 3.13 which leads to Burstall's twistor lift.

Suppose now that there is a  $t \in \mathbb{N}$  such that  $(P_0 W)_{(t)} = \mathbb{C}^n$ , equivalently,  $W_{(t)} = \underline{\mathcal{H}}_+$ ; such  $t$  exists if and only if  $P_0 W$  is full. Then  $\lambda^t \mathcal{H}_+ = \lambda^t W_{(t)} \subset W$  so that  $r \leq t$  and (5.8) extends to a *uniton factorization*:

$$W = W_{(0)} \subset W_{(1)} \subset \cdots \subset W_{(r)} \subset \cdots \subset W_{(t)} = \mathcal{H}_+;$$



this is the *factorization by  $A_z$ -images* of [36]. Note, however, that the extra terms  $W_{(i)}$  ( $i > r$ ) do not lengthen the associated  $F$ -filtration since, when  $i > r$ , we have  $\lambda^i W_{(i)} \subset \lambda W$  so that  $Z_i = 0$ .

Let  $\varphi : M \rightarrow U(n)$  be a harmonic map. Let  $(Z_i)$  be an  $A_z^\varphi$ -filtration of length  $t$ . Then each subbundle  $Z_i$  is a uniton for  $\varphi$ ; further  $Z_1$  is antibasic, and  $Z_t$  is basic. We have the following converse. In the proof, for any map  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  and integer  $i > 0$  we set  $f^0 = \text{identity map}$  and write  $f^i$  for the composition  $f^i = f \circ \dots \circ f$  (with  $i$  factors); further, for any subset  $V$  of  $\mathbb{C}^n$ , we write  $f^{-i}(V) = (f^i)^{-1}(V) = \{x \in \mathbb{C}^n : f^i(x) \in V\}$ .

**Proposition 5.7.** *Let  $\varphi : M \rightarrow U(n)$  be a nilconformal harmonic map, and let  $\alpha$  be a uniton for  $\varphi$ . Then we can find a strict  $A_z^\varphi$ -filtration  $(Z_i)$  with  $Z_k = \alpha$  for some  $k$ .*

*Proof.* Choose  $k \in \mathbb{N}$  such that  $(A_z^\varphi)^{-k}(\alpha) = \mathbb{C}^n$ ; since  $A_z^\varphi$  is the adjoint of  $-A_z^\varphi$ , this is equivalent to  $(A_z^\varphi)^k(\alpha^\perp) = \underline{0}$ . Then set  $Z_i = (A_z^\varphi)^{i-k}(\alpha)$ ; equivalently,  $Z_i^\perp = (A_z^\varphi)^{k-i}(\alpha^\perp)$ .  $\square$

Note the duality in these formulae, cf. Example 6.3. Note further that, if  $\alpha$  is antibasic, we can take  $k = 1$  so that  $Z_1 = \alpha$ ; if  $\alpha$  is basic, we have  $Z_k = \alpha$  and  $Z_{k+1} = \underline{0}$ . In the extreme case  $\alpha = \mathbb{C}^n$ , the formula gives the filtration  $Z_i = \text{Im}(A_z^\varphi)^i$  of Example 3.13, and when  $\alpha = \underline{0}$ , it gives the dual filtration  $Z_i = \ker(A_z^\varphi)^{k+1-i}$  of Example 3.14.

If  $\varphi$  has image in a Grassmannian and  $\alpha$  is a uniton for  $\varphi$ , then, as before, we say that  $\alpha$  *splits* (for  $\varphi$ ) if  $\alpha = \alpha \cap \varphi \oplus \alpha \cap \varphi^\perp$ . We deduce the following from Proposition 5.7.

**Theorem 5.8.** *Let  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  be a nilconformal harmonic map from a surface to a complex Grassmannian, and let  $\alpha$  be a uniton for  $\varphi$  which splits. Then there is a moving flag  $\psi = (\psi_0, \psi_1, \dots, \psi_t)$  with the uniton given by a sum  $\sum_{j=j_0}^t \psi_j$  of legs  $\psi_i$ , and a  $J_2$ -holomorphic twistor lift  $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_1, \dots, \tilde{\psi}_s)$  of  $\pm\varphi$  with each  $\tilde{\psi}$  the sum of some of the legs  $\psi_i$ .*

*Proof.* Proposition 5.7 gives an  $A_z^\varphi$ -filtration  $(Z_i)$  with one of the  $Z_i$  equal to  $\alpha$ . Since  $\alpha$  splits, it is clear that  $(Z_i)$  splits. By Proposition 3.11, we obtain a twistor lift  $\psi$  as described.  $\square$

**5.3.  $S^1$ -invariant maps and superhorizontal lifts.** We now consider an important special case of the above constructions when the twistor lifts are holomorphic with respect to both the non-integrable almost complex structure  $J_2$  and the integrable complex structure  $J_1$  of §3.1.

Let  $\Phi$  be a polynomial extended solution; denote its degree by  $r$ . Then  $\Phi$  is called  *$S^1$ -invariant* if  $\Phi_\lambda \Phi_\mu = \Phi_{\lambda\mu}$ . This clearly implies that  $\Phi$  is  $\nu$ -invariant so that  $\varphi = \Phi_{-1}$  is a harmonic map into a Grassmannian. Uhlenbeck showed [34, §10] that  $\Phi$  is  *$S^1$ -invariant if and only if its Uhlenbeck unitons  $\gamma_1, \dots, \gamma_r$  (Example 5.5) are nested:  $\gamma_i \supset \gamma_{i+1}$* . In that case, the formula (5.7) reduces to  $Z_i = \gamma_{r+1-i}^\perp$ . Equally well, it follows from [33, Proposition 3.14] that  $\Phi$  is  *$S^1$ -invariant if and only if its Segal unitons  $\beta_1, \dots, \beta_r$  (Example 5.4) are nested:*

$$(5.9) \quad \underline{0} = \beta_0 \subset \beta_1 \subset \dots \subset \beta_r \subset \beta_{r+1} = \mathbb{C}^n,$$

in which case  $\beta_i = \gamma_{r+1-i}$ ; it follows that  $\Phi$  and  $W = \Phi \mathcal{H}_+$  are given by

$$(5.10) \quad \Phi = \sum_{i=0}^r \lambda^i \pi_{\psi_i} \quad \text{and} \quad W = \sum_{i=0}^{r-1} \lambda^i \beta_{i+1} + \lambda^r \underline{\mathcal{H}}_+,$$

giving a harmonic map  $\varphi = \Phi_{-1} = \sum_{k=0}^{[r/2]} \psi_{2k}$  where  $\psi_i = \beta_{i+1} \ominus \beta_i$ .

A nested sequence (5.9) of subbundles of  $\mathbb{C}^n$  is called *superhorizontal* (cf. [6]) if (i) each subbundle is holomorphic and (ii)  $\partial_z$  maps sections of  $\beta_i$  into  $\beta_{i+1}$ . Then (see, for example, [33, Proposition 3.14]), *if  $\Phi$  is  $S^1$ -invariant, the sequence  $(\beta_i)$  of its Segal unitons is superhorizontal.*

As usual, set  $d_i = \text{rank } \psi_i$ ; the  $d_i$  are all non-zero if and only if  $\Phi$  is normalized, in which case the canonical  $J_2$ -holomorphic lift  $\psi = (\psi_0, \psi_1, \dots, \psi_r) : M \rightarrow F_{d_0, \dots, d_r}$  of  $\varphi = \Phi_{-1}$  defined by  $\Phi$  is given by  $\psi_i = Z_i \ominus Z_{i+1} = \beta_{i+1} \ominus \beta_i$ . Superhorizontality of the sequence  $(\beta_i)$  can be interpreted as saying that the derivative of  $\psi$  lies in the *superhorizontal distribution*, by which we mean the subbundle of the  $(1,0)$ -horizontal bundle given by  $\sum_{i=0}^{r-1} \text{Hom}(\psi_i, \psi_{i+1})$ , in which case  $\psi$  is horizontal and both  $J_1$ - and  $J_2$ -holomorphic. Thus, *let  $\Phi$  be a normalized extended solution. Then the canonical lift of  $\varphi = \Phi_{-1}$  defined by  $\Phi$  is superhorizontal if and only if  $\Phi$  is  $S^1$ -invariant.*

**Remark 5.9.** (i) When  $r = 1$ , superhorizontality is automatic; when  $r = 2$ , superhorizontality is equivalent to horizontality.

(ii) Given a harmonic map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$ , there is a superhorizontal holomorphic lift of  $\varphi$  or  $\varphi^\perp$  if and only if there is an  $S^1$ -invariant polynomial extended solution  $\Phi$  with  $\Phi_{-1} = \pm\varphi$ . Indeed, given such a lift,  $\Phi$  is given by (5.10); conversely, after normalizing  $\Phi$ , the canonical lift is superhorizontal as explained above.

(iii) A harmonic map  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  which has a superhorizontal holomorphic lift is called *isotropic* in [18] where it is characterized geometrically, see also [13].

**Example 5.10.** A *superhorizontal sequence of length one* is just a single holomorphic subbundle  $\beta_1 \subset \underline{\mathbb{C}}^n$ . The corresponding extended solution (5.10) is given by  $\Phi = \pi_{\beta_1} + \lambda\pi_{\beta_1}^\perp$  and  $W = \beta_1 + \lambda\underline{\mathcal{H}}_+$ . Set  $d_0 = \text{rank } \beta_1$ . Then the resulting harmonic map  $\varphi = \Phi_{-1} : M \rightarrow G_{d_0}(\mathbb{C}^n)$  is holomorphic and  $-\Phi_{-1} = \varphi^\perp : M \rightarrow G_{n-d_0}(\mathbb{C}^n)$  is antiholomorphic; all holomorphic and antiholomorphic maps  $M \rightarrow G_*(\mathbb{C}^n)$  are obtained in this way. The canonical lift of  $\varphi$  defined by  $\Phi$  is  $\psi = (\psi_0, \psi_1) = (\beta_1, \beta_1^\perp) : M \rightarrow F_{d_0, n-d_0}$ . The projection  $\pi_e : F_{d_0, n-d_0} \rightarrow G_{d_0}(\mathbb{C}^n)$  is given by  $(\psi_0, \psi_1) \mapsto \psi_0$  and is bijective.

For the next examples, as before, for any  $i \in \mathbb{N}$  and holomorphic map  $f : M \rightarrow G_*(\mathbb{C}^n)$ , equivalently, holomorphic subbundle of  $\underline{\mathbb{C}}^n$ , we denote by  $f_{(i)}$  the  $i$ th osculating space spanned by derivatives of local holomorphic sections of  $f$  up to order  $i$ .

**Example 5.11.** (i) A *superhorizontal sequence of length 2* is a nested pair  $\beta_1 \subset \beta_2$  of holomorphic subbundles of  $\underline{\mathbb{C}}^n$  with  $\partial_z(\Gamma(\beta_1)) \subset \Gamma(\beta_2)$ ; such a pair is called a  $\partial'$ -pair in [17]. Equivalently,  $(\beta_1, \beta_2^\perp)$  is a *mixed pair* in the sense of [9] (generalized to subbundles of arbitrary rank), i.e.,  $\beta_1$  is a holomorphic subbundle of  $\underline{\mathbb{C}}^n$ ,  $\beta_2^\perp$  is an antiholomorphic one, and  $\partial_z(\Gamma(\beta_1))$  has values perpendicular to  $\beta_2^\perp$ . The corresponding extended solutions  $\Phi$  and  $W = \Phi\underline{\mathcal{H}}_+$  of (5.10) are given by

$$(5.11) \quad \Phi = \pi_{\beta_1} + \lambda\pi_\varphi + \lambda^2\pi_{\beta_2}^\perp \quad \text{and} \quad W = \beta_1 + \lambda\beta_2 + \lambda^2\underline{\mathcal{H}}_+.$$

The resulting harmonic map  $\varphi = \Phi_{-1}$  is given by  $\varphi = \beta_1 \oplus \beta_2^\perp$ ; it is also called a *mixed pair*. Its orthogonal complement is the harmonic map  $\varphi^\perp = \beta_2 \ominus \beta_1$ ; this is *strongly isotropic* [17, (1.6)] in the sense that the Gauss transforms (see §2.2)  $G^{(i)}(\varphi^\perp)$  and  $G^{(j)}(\varphi^\perp)$  are orthogonal for all integers  $i \neq j$ . All strongly isotropic harmonic maps  $M \rightarrow G_*(\mathbb{C}^n)$  are obtained from a  $\partial'$ -pair  $\beta_1 \subset \beta_2$  in this way [17, §4]; indeed, we may take  $\beta_1 = \sum_{i < 0} G^{(i)}(\varphi^\perp)$  and  $\beta_2^\perp = \sum_{i > 0} G^{(i)}(\varphi^\perp)$ . The canonical lift of  $\varphi$  defined by the extended solution (5.11) is given by  $\psi = (\psi_0, \psi_1, \psi_2) = (\beta_1, \varphi^\perp, \beta_2^\perp) : M \rightarrow F_{d_0, d_1, d_2}$  where  $d_i = \text{rank } \psi_i$ . This is, of course, superhorizontal.

Note that a strongly isotropic map is certainly strongly conformal, and the lift  $\psi$  just defined is of the type described in Example 3.15(iii). For a map  $\varphi : M \rightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ , the notion of strong isotropy reduces to the notion of (complex) isotropy as used in [16].

(ii) Let  $f : M \rightarrow \mathbb{C}P^{n-1}$  be a full holomorphic map and let  $i \in \{1, 2, \dots, n-1\}$ . Setting  $\beta_1 = f_{(i-1)}$  and  $\beta_2 = f_{(i)}$  gives a  $\partial'$ -pair  $\beta_1 \subset \beta_2$ , and so a full isotropic harmonic map  $\varphi^\perp : M \rightarrow \mathbb{C}P^{n-1}$  given by  $\varphi^\perp = G^{(i)}(f)$ . All isotropic harmonic maps  $M \rightarrow \mathbb{C}P^{n-1}$  are given this way and so all harmonic maps from  $S^2$  to  $\mathbb{C}P^{n-1}$ , see [16]; holomorphic and antiholomorphic maps are given by the extreme cases  $i = 1$  and  $i = n-1$ , respectively. Excluding those cases, as in part (i), the canonical lift of  $\varphi$  defined by the extended solution (5.11) is  $\psi = (\beta_1, \varphi^\perp, \beta_2^\perp)$ ; this is a superhorizontal holomorphic lift of  $\varphi$ . Note that  $\varphi^\perp$  is strongly conformal, but  $\varphi$  is not. In fact  $\varphi^\perp : M \rightarrow \mathbb{C}P^{n-1}$  can have no twistor lift  $\psi$  to a flag manifold; indeed, such a twistor lift  $\psi = (\psi_0, \psi_1, \psi_2, \dots)$  would have to have at least three legs, but then  $\varphi^\perp = \pi \circ \psi$  would contain  $\psi_0 \oplus \psi_2$ , which has rank at least two.

(iii) Let  $f : M \rightarrow \mathbb{C}P^{n-1}$  be a full holomorphic map and let  $i \in \{2, 3, \dots, n-3\}$ . Setting  $\beta_1 = f_{(i-2)}$  and  $\beta_2 = f_{(i)}$  gives a  $\partial'$ -pair  $\beta_1 \subset \beta_2$  and so a strongly isotropic harmonic map  $\varphi^\perp = G^{(i-1)}(f) \oplus G^{(i)}(f)$ ; such a harmonic map is called a *Frenet pair* [9]. The canonical lift of  $\varphi$  defined by (5.10) is again  $\psi = (\beta_1, \varphi^\perp, \beta_2^\perp)$ .

In contrast to part (ii),  $\varphi$  is strongly conformal, and Example 3.15(iv) provides the unique  $J_2$ -holomorphic lift of  $\varphi^\perp$  with three legs:  $\psi = (G''(\varphi), \varphi, G'(\varphi)) = (G^{(i)}(f), \varphi, G^{(i-1)}(f))$ .

6. TWISTOR LIFTS OF MAPS INTO REAL GRASSMANNIANS AND  $O(2m)/U(m)$ 

**6.1. Twistor spaces.** We now consider the symmetric spaces of the *orthogonal group*  $O(n)$  and its identity component, the *special orthogonal group*  $SO(n)$ . We think of  $O(n)$  as the totally geodesic submanifold  $\{g \in U(n) : g = \bar{g}\}$  of  $U(n)$ . For each  $k$ , the *real Grassmannian*  $G_k(\mathbb{R}^n) = O(n)/O(k) \times O(n-k)$  is a symmetric space; it may be thought of as the totally geodesic submanifold  $\{V \in G_k(\mathbb{C}^n) : V = \bar{V}\}$  of  $G_k(\mathbb{C}^n)$ , and, via the Cartan embedding (2.2), as a totally geodesic submanifold of the orthogonal group  $O(n)$ , and so also of  $U(n)$ . We now identify twistor spaces for the real Grassmannian as subspaces of those for  $G_k(\mathbb{C}^n)$ .

Let  $n, d_0, d_1, \dots, d_s$  be positive integers with  $d_s + 2 \sum_{i=0}^{s-1} d_i = n$ . Set  $d_i = d_{2s-i}$  for  $i = s+1, \dots, 2s$  so that  $\sum_{i=0}^{2s} d_i = n$ , and define a submanifold of the flag manifold  $F_{d_0, \dots, d_{2s}}$  of §3.1 by

$$F_{d_0, \dots, d_s}^{\mathbb{R}} = \{\psi = (\psi_0, \psi_1, \dots, \psi_{2s}) \in F_{d_0, \dots, d_{2s}} : \psi_i = \bar{\psi}_{2s-i} \ \forall i\}$$

(here, by  $\bar{\psi}_{2s-i}$  we mean the complex conjugate  $\overline{\psi_{2s-i}}$ ). Note that the middle leg  $\psi_s$  is real, i.e.,  $\bar{\psi}_s = \psi_s$ . Further note that  $F_{d_0, \dots, d_s}^{\mathbb{R}}$  is a complex submanifold of  $F_{d_0, \dots, d_{2s}}$  with respect to the complex structures  $J_1$  and  $J_2$ . Hence the twistor fibration (3.2) restricts to a twistor fibration  $\pi_e^{\mathbb{R}} : F_{d_0, \dots, d_s}^{\mathbb{R}} \rightarrow G_k(\mathbb{R}^n)$  where, as before,  $k = \sum_{j=0}^s d_{2j}$  and  $\pi_e^{\mathbb{R}}(\psi) = \sum_{j=0}^s \psi_{2j}$ .

On using  $\psi_i = \bar{\psi}_{2s-i}$ , these can be written in terms of just  $(\psi_0, \psi_1, \dots, \psi_s)$  as follows.

$$(6.1) \quad \begin{cases} k = 2 \sum_{k=0}^{s/2-1} d_{2k} + d_s, & \pi_e^{\mathbb{R}}(\psi) = \sum_{k=0}^{s/2-1} (\psi_{2k} \oplus \bar{\psi}_{2k}) \oplus \psi_s \quad (s \text{ even}); \\ k = 2 \sum_{j=0}^{(s-1)/2} d_{2j}, & \pi_e^{\mathbb{R}}(\psi) = \sum_{j=0}^{(s-1)/2} (\psi_{2j} \oplus \bar{\psi}_{2j}) \quad (s \text{ odd}). \end{cases}$$

Note that if  $s$  is even,  $n-k$  is even and if  $s$  is odd,  $k$  is even; further, from (6.1), we have  $s \leq \min(k-1, n-k)$ .

As a homogeneous space,  $F_{d_0, \dots, d_s}^{\mathbb{R}} = O(n)/H$  where  $H = U(d_0) \times \dots \times U(d_{s-1}) \times O(d_s)$ . Write  $H = H_1 \times H_2$  where  $H_1 = \{\prod_{j=0}^{s/2-1} U(d_{2j})\} \times O(d_s)$  if  $s$  is even and  $\prod_{j=0}^{(s-1)/2} U(d_{2j})$  if  $s$  is odd. Then the projection  $\pi_e^{\mathbb{R}}$  is the homogeneous projection  $O(n)/H \rightarrow O(n)/O(k) \times O(n-k)$  induced by the inclusion of  $H = H_1 \times H_2$  in  $O(k) \times O(n-k)$  given by the canonical inclusions of  $H_1$  in  $O(k)$  and  $H_2$  in  $O(n-k)$ .

We can also write  $F_{d_0, \dots, d_s}^{\mathbb{R}} = SO(n)/\tilde{H}$  where  $\tilde{H} = U(d_0) \times \dots \times U(d_{s-1}) \times SO(d_s)$ . The Grassmannian  $G_k(\mathbb{R}^n)$  is double-covered by the Grassmannian  $\tilde{G}_k(\mathbb{R}^n)$  of *oriented*  $k$ -dimensional subspaces of  $\mathbb{R}^n$  and the projection  $\pi_e^{\mathbb{R}}$  lifts to a projection  $\tilde{\pi}_e^{\mathbb{R}} : F_{d_0, \dots, d_s}^{\mathbb{R}} = SO(n)/\tilde{H} \rightarrow SO(n)/SO(k) \times SO(n-k) = \tilde{G}_k(\mathbb{R}^n)$ , providing a twistor space for  $\tilde{G}_k(\mathbb{R}^n)$ . We have a double covering  $\tilde{G}_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$  which forgets orientation; composing that with the inclusion map  $G_k(\mathbb{R}^n) \hookrightarrow G_k(\mathbb{C}^n)$  gives a canonical totally geodesic isometric immersion of  $\tilde{G}_k(\mathbb{R}^n)$  into  $G_k(\mathbb{C}^n)$ .

**Example 6.1.** For any  $m \in \{1, 2, \dots\}$ , the mapping  $(V, Y, \bar{V}) \mapsto V$  identifies the space  $F_{m,1}^{\mathbb{R}} = O(2m+1)/U(m) \times O(1) \cong SO(2m+1)/U(m)$  with the space of isotropic subspaces of  $\mathbb{C}^{2m+1}$  of dimension  $m$ ; the bundle  $\pi_e^{\mathbb{R}} : F_{m,1}^{\mathbb{R}} \rightarrow \mathbb{R}P^{2m}$  with  $\pi_e(V, Y, \bar{V}) = Y$  can be identified with the bundle  $Z \rightarrow \mathbb{R}P^{2m}$  of almost Hermitian structures (cf. Remark 3.2(ii)), and it lifts to a fibre bundle  $\tilde{\pi}_e^{\mathbb{R}} : F_{m,1}^{\mathbb{R}} \rightarrow S^{2m}$  which is the bundle  $Z^+ \rightarrow S^{2m}$  of *positive* almost Hermitian structures on  $S^{2m}$ . In particular, the double covering  $Sp(2) \rightarrow SO(5)$  maps  $U(1) \times Sp(1)$  to  $U(2)$ , showing that  $F_{2,1}^{\mathbb{R}} \cong Sp(2)/U(1) \times Sp(1) \cong \mathbb{C}P^3$ , hence the fibration  $F_{2,1}^{\mathbb{R}} \rightarrow S^4$  is the classical twistor fibration  $\mathbb{C}P^3 \rightarrow S^4$ .

The orthogonal group has another symmetric space,  $O(2m)/U(m)$ , the space of *orthogonal complex structures*; this has identity component,  $SO(2m)/U(m)$ , the space of *positive orthogonal complex structures*. By mapping a complex structure to its  $(-i)$ -eigenspace,  $O(2m)/U(m)$  may be identified with the totally geodesic submanifold  $\{V \in G_m(\mathbb{C}^{2m}) : V = \bar{V}^\perp\}$ . This in turn may be identified via the Cartan embedding, with a totally geodesic submanifold of  $iO(n)$  where  $iO(n)$  denotes the totally geodesic submanifold  $\{ig : g \in O(n)\} = \{g \in U(n) : g = -\bar{g}\}$  of  $U(n)$ .

We obtain twistor spaces for  $O(2m)/U(m)$  as restrictions of those for  $G_m(\mathbb{C}^{2m})$  as follows. Let  $m, d_0, d_1, \dots, d_s$  be positive integers with  $m = d_0 + \dots + d_s$  and set  $d_i = d_{2s+1-i}$  for  $i =$

$s + 1, \dots, 2s + 1$ . Let  $\mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}}$  be the submanifold of  $F_{d_0, \dots, d_{2s+1}}$  given by

$$\mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}} = \{\psi = (\psi_0, \psi_1, \dots, \psi_{2s+1}) \in F_{d_0, \dots, d_{2s+1}} : \psi_i = \overline{\psi}_{2s+1-i} \ \forall i\}.$$

This is a complex submanifold with respect to the almost complex structures  $J_1$  and  $J_2$  of  $F_{d_0, \dots, d_{2s+1}}$ , so the projection (3.2) restricts to a twistor fibration  $\pi_e^{\mathbb{R}} : \mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}} \rightarrow \mathrm{O}(2m)/\mathrm{U}(m)$ .

As a homogeneous space,  $\mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}} = \mathrm{O}(2m)/\mathrm{U}(d_0) \times \dots \times \mathrm{U}(d_s)$ , and  $\pi_e^{\mathbb{R}}$  is the homogeneous projection  $\mathrm{O}(2m)/\mathrm{U}(d_0) \times \dots \times \mathrm{U}(d_s) \rightarrow \mathrm{O}(2m)/\mathrm{U}(m)$  given by the canonical inclusion of the product  $\mathrm{U}(d_0) \times \dots \times \mathrm{U}(d_s)$  in  $\mathrm{U}(m)$ . This restricts to a twistor fibration  $\tilde{\pi}_e^{\mathbb{R}} : \mathrm{SO}(2m)/\mathrm{U}(d_0) \times \dots \times \mathrm{U}(d_s) \rightarrow \mathrm{SO}(2m)/\mathrm{U}(m)$ .

**6.2. Some involutions.** We describe some involutions on our various types of filtrations; real cases will then appear as their fixed points. For a map  $\varphi : M \rightarrow \mathrm{U}(n)$ ,  $\overline{\varphi}$  will denote its complex conjugate; thus  $\varphi = \overline{\varphi}$  (resp.  $\varphi = -\overline{\varphi}$ ) if and only if  $\varphi$  is real, i.e., has image in  $\mathrm{O}(n)$  (resp.  $\varphi$  has image in  $i\mathrm{O}(n)$ , equivalently  $i\varphi$  is real).

**Lemma 6.2.** *Let  $\varphi : M \rightarrow \mathrm{U}(n)$  be a nilconformal harmonic map.*

(i) *Let  $(Z_i)$  be an  $A_z^\varphi$ -filtration of length  $t$ ; denote its legs by  $\psi_i = Z_i \ominus Z_{i+1}$ . Set*

$$(6.2) \quad \tilde{Z}_i = \overline{Z}_{t+1-i}^\perp \quad (i = 0, 1, \dots, t+1).$$

*Then  $(\tilde{Z}_i)$  is an  $A_z^{\overline{\varphi}}$ -filtration of the same length with legs  $\tilde{\psi}_i = \tilde{Z}_i \ominus \tilde{Z}_{i+1}$  given by  $\tilde{\psi}_i = \overline{\psi}_{t-i}$ .*

*Further, if  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  and  $(Z_i)$  is split (resp. alternating) for  $\varphi$ , then so is  $(\tilde{Z}_i)$ .*

(ii) *If  $\varphi$  or  $i\varphi$  is real, then  $(Z_i) \mapsto (\tilde{Z}_i)$  defines an involution on the set of  $A_z^\varphi$ -filtrations.*

*Proof.* Since the adjoint of  $A_z^\varphi$  is  $-A_z^{\overline{\varphi}}$ , the condition  $A_z^\varphi(Z_i) \subset Z_{i+1}$  is equivalent to  $A_z^{\overline{\varphi}}(Z_{i+1}^\perp) \subset Z_i^\perp$ , and this is equivalent to  $A_z^{\overline{\varphi}}(\tilde{Z}_{t-i}) \subset \tilde{Z}_{t+1-i}$ . Similarly  $D_z^\varphi(\Gamma(Z_i)) \subset \Gamma(Z_i)$  is equivalent to  $D_z^{\overline{\varphi}}(\Gamma(\tilde{Z}_{t+1-i})) \subset \Gamma(\tilde{Z}_{t+1-i})$ ; the rest is clear.  $\square$

**Example 6.3.** (i) If  $Z_i = \mathrm{Im}(A_z^\varphi)^i$  as in Example 3.13, then  $\tilde{Z}_i = \ker(A_z^{\overline{\varphi}})^{t+1-i}$ . Replacing  $\varphi$  by  $\overline{\varphi}$  gives Example 3.14. A similar process gives Example 3.15(ii) from Example 3.15(i).

(ii) An equivalent conclusion to part (i) of the lemma is that  $\tilde{Z}_i = Z_{t+1-i}^\perp$  defines an  $A_z^\varphi$ -filtration with respect to the conjugate complex structure on  $M$ .

We have a corresponding involution of  $F$ -filtrations as follows. For a map  $\Phi : M \rightarrow \Omega\mathrm{U}(n)$ , set  $W = \Phi\mathcal{H}_+$  and  $\varphi = \Phi_{-1}$ . For a fixed integer  $r$ , write  $W^I = \lambda^{r-1}\overline{W}^\perp$ , then  $W^I = \tilde{\Phi}\mathcal{H}_+$  where  $\tilde{\Phi} = \lambda^r\overline{\Phi}$  (cf. [33, Remark 2.7]). If  $W$  is an extended solution, it is easily checked that  $W^I$  is also an extended solution. Writing  $\tilde{\varphi} = \tilde{\Phi}_{-1}$ , we have  $\tilde{\varphi} = (-1)^r\varphi$ .

**Definition 6.4.** Let  $r \in \mathbb{Z}$ . We call  $W$  or  $\Phi$  *real of degree  $r$*  if  $W^I = W$ , equivalently  $\Phi = \lambda^r\overline{\Phi}$ .

If  $r = 2s$  is even, this says that  $\lambda^{-s}\Phi$  has values in  $\Omega\mathrm{O}(n)$ , and implies that  $\varphi$  is real, i.e., has values in  $\mathrm{O}(n)$ . When  $r$  is odd, it implies that  $i\varphi$  is real, see §6.4 for the application of that case.

**Lemma 6.5.** *Let  $\Phi : M \rightarrow \Omega\mathrm{U}(n)$  be a polynomial extended solution of degree  $r$  with  $\Phi_{-1} = \varphi$ . Set  $W = \Phi\mathcal{H}_+$ . Let  $(Y_i)$  be an  $F$ -filtration of  $W$ ; denote its length by  $t$ . Set*

$$(6.3) \quad \tilde{Y}_i = \lambda^r \overline{Y}_{t+1-i}^\perp \quad (i = 0, 1, \dots, t+1).$$

*Then*

- (i)  $(\tilde{Y}_i)$  is an  $F$ -filtration of  $W^I$ ;
- (ii)  $\tilde{Y}_i = Y_i$  for all  $i$ ;
- (iii) set  $Z_i = P_0 \circ \Phi^{-1}(Y_i)$  and  $\tilde{Z}_i = P_0 \circ \tilde{\Phi}^{-1}(\tilde{Y}_i)$ . Then  $Z_i$  and  $\tilde{Z}_i$  are related by (6.2);
- (iv) if  $\Phi$  is real of degree  $r$ , then  $(Y_i) \mapsto (\tilde{Y}_i)$  defines an involution on the set of  $F$ -filtrations of  $W$ .

*Proof.* (i) The condition  $F(\Gamma(Y_i)) \subset \Gamma(Y_{i+1})$  is clearly equivalent to  $F(\Gamma(\bar{Y}_{i+1}^\perp)) \subset \Gamma(\bar{Y}_i^\perp)$ . Also  $\tilde{Y}_0 = \lambda^r \bar{Y}_{t+1}^\perp = \lambda^{r-1} \bar{W}^\perp = W^I$  and  $\tilde{Y}_{t+1} = \lambda^r \bar{Y}_0^\perp = \lambda^r \bar{W}^\perp = \lambda W^I$ .

(ii) This is quickly checked.

(iii) Noting that  $Z_i = P_0 \circ \Phi^{-1} Y_i$  is equivalent to  $\Phi^{-1} Y_i + \lambda \underline{H}_+ = Z_i + \lambda \underline{H}_+$ , we have

$$\tilde{Z}_i + \lambda \underline{H}_+ = \tilde{\Phi}^{-1} \tilde{Y}_i + \lambda \underline{H}_+ = \bar{\Phi}^{-1} \bar{Y}_{t+1-i}^\perp + \lambda \underline{H}_+ = \overline{\Phi^{-1} Y_{t+1-i}}^\perp + \lambda \underline{H}_+ = \bar{Z}_{t+1-i}^\perp + \lambda \underline{H}_+,$$

giving the result.

(iv) Immediate from (i).  $\square$

Now let  $\Phi$  be a polynomial extended solutions of degree at most  $r$ . Then the involution  $W \mapsto W^I$  above gives another polynomial extended solution  $\tilde{\Phi} = \lambda^r \bar{\Phi}$  of degree at most  $r$ , see [33, Example 3.8]. We now see what this involution does to the canonical filtration.

**Proposition 6.6.** *Let  $\Phi$  a polynomial extended solution of degree at most  $r$ .*

(i) *Let  $(Y_i)$  be the canonical  $F$ -filtration of  $W$ , and define  $(\tilde{Y}_i)$  by (6.3) with  $t = r$ ; then  $(\tilde{Y}_i)$  is the canonical  $F$ -filtration of  $W^I$ .*

(ii) *If  $\Phi$  is real of degree  $r$ , then (a) the canonical  $F$ -filtration  $(Y_i)$  is real, i.e., fixed under the involution (6.3); (b) the canonical  $A_z^\varphi$ -filtration  $Z_i = P_0 \circ \Phi^{-1}(Y_i)$  is real, i.e., fixed under the involution (6.2); (c) the legs  $\psi_i = Z_i \ominus Z_{i+1}$  satisfy  $\psi_i = \bar{\psi}_{r-i}$  ( $i = 0, 1, \dots, r$ ).*

*Proof.* (i) We have  $Y_i = W \cap \lambda^i \underline{H}_+ + \lambda W$  so that

$$\begin{aligned} \tilde{Y}_i &= \lambda^r \bar{Y}_{r-i+1}^\perp = \lambda^r \{(\bar{W}^\perp + \lambda^{i-r} \underline{H}_+) \cap (\lambda^{-1} \bar{W}^\perp)\} \\ &= (\lambda W^I + \lambda^i \underline{H}_+) \cap W^I = W^I \cap \lambda^i \underline{H}_+ + \lambda W^I. \end{aligned}$$

(ii) Immediate from (i).  $\square$

**6.3.  $J_2$ -holomorphic lifts for maps to real Grassmannians.** To apply our work to harmonic maps into real Grassmannians, we need the following existence result for extended solutions.

**Proposition 6.7.** *Let  $\Phi : M \rightarrow \Omega U(n)$  be a  $\nu$ -invariant polynomial extended solution which is real of some even degree  $r = 2s$ . Then (i)  $\varphi = (-1)^s \Phi_{-1} : M \rightarrow G_k(\mathbb{R}^n)$  is a harmonic map of finite uniton number with  $n - k$  even; (ii) all such harmonic maps  $\varphi$  are given this way; in fact, we may take  $\Phi$  to be normalized of degree at most  $2 \min(k - 1, n - k)$ , if  $s$  even, and  $2 \min(k, n - k - 1)$ , if  $s$  is odd.*

*Proof.* (i) This is a consequence of the formula for  $\varphi$  in Proposition 4.6; the parity of  $n - k$  following from the symmetry of the legs as in Proposition 6.6(ii).

(ii) By [33, Lemma 6.6], there is a  $\nu$ -invariant extended solution  $\Psi : M \rightarrow \Omega O(n)$  of the form  $\Psi = \sum_{\ell=-s}^s \lambda^\ell T_\ell$  with  $T_{-\ell} = \bar{T}_\ell$ ,  $T_s \neq 0$  and  $\Psi_{-1} = \varphi$ . Setting  $\Phi = \lambda^s \Psi$  gives a  $\nu$ -invariant real polynomial extended solution of degree  $2s$  with  $\Phi_{-1} = (-1)^s \varphi$ . That we may take  $\Phi$  normalized with the given bounds on the degree follows from [33, Proposition 6.23].  $\square$

**Remark 6.8.** The statement (ii) is false without the factor  $(-1)^s$ , see Example 6.11(ii) below. Also, if  $n - k$  is odd, we may embed  $G_k(\mathbb{R}^n)$  in  $G_k(\mathbb{R}^{n+1})$ , then harmonic maps from  $M$  to  $G_k(\mathbb{R}^n)$  are obtained as non-full harmonic maps into  $G_k(\mathbb{R}^{n+1})$ .

**Theorem 6.9.** *Let  $\Phi : M \rightarrow \Omega U(n)$  be a  $\nu$ -invariant polynomial extended solution which is normalized and real of even degree  $r = 2s$ . Let  $\varphi = \Phi_{-1} : M \rightarrow G_k(\mathbb{R}^n)$  be the resulting harmonic map. Then  $\varphi$  has a  $J_2$ -holomorphic lift  $\psi : M \rightarrow F_{d_0, d_1, \dots, d_s}^{\mathbb{R}}$  for some  $(d_0, d_1, \dots, d_s)$  satisfying (6.1), namely the canonical twistor lift defined by  $\Phi$  (see Theorem 4.8).  $\square$*

On applying Proposition 6.7, we obtain the following corollary.

**Corollary 6.10.** *Let  $\varphi : M \rightarrow G_k(\mathbb{R}^n)$  be a harmonic map of finite uniton number. Then either  $\varphi$  or  $-\varphi$  ( $= \varphi^\perp$ ) has a  $J_2$ -holomorphic twistor lift  $\psi : M \rightarrow F_{d_0, d_1, \dots, d_s}^{\mathbb{R}}$  for some  $(d_0, d_1, \dots, d_s)$  satisfying (6.1), namely the canonical twistor lift defined by a normalized extended solution  $\Phi$  of  $\pm \varphi$ .  $\square$*

**Example 6.11.** (i) Recall from Example 5.11 that a  $\mathcal{D}$ -pair (i.e., superhorizontal sequence of length 2)  $\underline{0} \subset \beta_1 \subset \beta_2 \subset \mathbb{C}^n$  gives rise to two harmonic maps: the mixed pair  $\varphi = \beta_1 \oplus \beta_2^\perp$  and the strongly isotropic map  $\varphi^\perp = \beta_2 \ominus \beta_1$ . The harmonic maps  $\varphi$  and  $\varphi^\perp$  are real, i.e., have image in  $G_*(\mathbb{R}^n)$ , if and only if  $\beta_2^\perp = \overline{\beta}_1$ , in which case  $(\beta_1, \beta_2^\perp) = (\beta_1, \overline{\beta}_1)$ , and the resulting harmonic map  $\varphi = \beta_1 \oplus \overline{\beta}_1$ , is called a *real mixed pair* [1]. In this case, the canonical lift of  $\varphi$  defined by the extended solution (5.11) is the superhorizontal holomorphic map  $\psi = (\beta_1, \varphi^\perp, \overline{\beta}_1) : M \rightarrow F_{d_0, d_1}^{\mathbb{R}}$ , where  $d_0 = \text{rank } \beta_1$  and  $d_1 = n - 2d_0$ .

In the case that  $\beta_1$  has rank one, we have  $\varphi = \beta_1 \oplus \overline{\beta}_1 : M \rightarrow G_2(\mathbb{R}^n)$ . We may identify the twistor space  $F_{1, n-2}^{\mathbb{R}}$  of  $G_2(\mathbb{R}^n)$  with the quadric  $Q_{n-2} = \{L = [L_1, \dots, L_n] \in \mathbb{C}P^{n-1} : \sum_1^n L_i^2 = 0\}$  via the map  $(\psi_0, \psi_1, \psi_2) \mapsto \psi_0$ , then  $\pi_e^{\mathbb{R}} : Q_{n-2} \rightarrow G_2(\mathbb{R}^n)$  is the double cover  $L \mapsto L \oplus \overline{L}$ . The canonical lift of  $\varphi$  is the superhorizontal holomorphic map  $\beta_1 \cong (\beta_1, \varphi^\perp, \overline{\beta}_1) : M \rightarrow Q_{n-2} \cong F_{1, n-2}^{\mathbb{R}}$ . See [1] for more information on harmonic maps from a surface to  $G_2(\mathbb{R}^n)$ .

(ii) Let  $f : M \rightarrow \mathbb{C}P^{n-1}$  be a full holomorphic map which is *totally isotropic* [16] in the sense that  $G^{(n-1)}(f) = \overline{f}$ . Then  $n - 1$  is even, say  $2m$ , and  $f_{(m-1)}$  is a maximal isotropic subbundle of  $\mathbb{C}^n$ . Setting  $\beta_1 = f_{(m-1)}$  and  $\beta_2 = f_{(m)}$ , we have  $\beta_2^\perp = \overline{\beta}_1$ , so we obtain a real mixed pair  $\varphi = \beta_1 \oplus \overline{\beta}_1 : M \rightarrow G_{2m}(\mathbb{R}^{2m+1})$  with  $\varphi^\perp : M \rightarrow \mathbb{R}P^{2m}$  a full harmonic map whose composition with the canonical inclusion of  $\mathbb{R}P^{2m}$  in  $\mathbb{C}P^{2m}$  is isotropic. E. Calabi and S.-S. Chern showed (see [16]) that all such isotropic harmonic maps from a surface to a real projective space  $\mathbb{R}P^{n-1}$ , in particular, all harmonic maps from the 2-sphere, are given this way; all harmonic maps from  $S^2$  to a sphere  $S^{n-1}$  can be obtained as double covers of those maps. The canonical lift of  $\varphi$  defined by (5.11) is the superhorizontal holomorphic map  $\psi = (\beta_1, \varphi^\perp, \overline{\beta}_1) : M \rightarrow F_{m, 1}^{\mathbb{R}} = \text{O}(2m+1)/(\text{U}(m) \times \text{O}(1)) = \text{SO}(2m+1)/\text{U}(m)$ .

As before,  $\varphi^\perp$  has no twistor lift, as it would have to be the sum of at least two legs.

(iii) Example 5.11 part (iii) does not specialize to give real maps; indeed *there are no real Frenet pairs* [1, Prop. 5.10].

(iv) Generalizing part (i), let  $\underline{0} = \beta_0 \subset \beta_1 \subset \dots \subset \beta_r \subset \beta_{r+1} = \mathbb{C}^n$  be a nested sequence of subbundles which is superhorizontal (see §5.3). Recall that such a sequence defines an  $S^1$ -invariant extended solution  $\Phi$  which is polynomial of degree  $r$ , and a harmonic map  $\varphi = \Phi_{-1}$  given by (5.10). Say that a nested sequence  $(\beta_i)$  is *real (of degree  $r$ )* if  $\beta_i^\perp = \overline{\beta}_{r+1-i}$  for all  $i$ ; on setting  $\psi_i = \beta_{i+1} \ominus \beta_i$  ( $i = 0, 1, \dots, r$ ), this is equivalent to  $\psi_i = \overline{\psi}_{r-i}$ . A *superhorizontal sequence  $(\beta_i)$  is real if and only if the corresponding extended solution (5.10) is real*. Now suppose that  $(\beta_i)$  is real of even degree  $r = 2s$ . Then  $\varphi = \sum_j \psi_{2j}$  is a map into  $G_*(\mathbb{R}^n)$  and the canonical  $J_2$ -holomorphic twistor lift of  $\varphi : M \rightarrow G_*(\mathbb{R}^n)$  defined by  $\Phi$  is  $\psi = (\psi_0, \psi_1, \dots, \psi_r) : M \rightarrow F_{d_0, \dots, d_r}^{\mathbb{R}}$ . As in §5.3, it is superhorizontal.

We now see how to obtain twistor lifts for real nilconformal maps by a method which extends that which gave the Burstall lift of Example 3.13. It is easy to check (in fact, it is a special case of Lemma 6.2(ii)) that, *if  $\varphi$  or  $i\varphi$  is real and  $\alpha$  is a uniton for  $\varphi$ , so is  $\overline{\alpha}^\perp$* . Note that  $\alpha$  is isotropic if and only if  $\overline{\alpha}^\perp \subset \alpha$ , and *maximally* isotropic exactly when  $\overline{\alpha}^\perp = \alpha$ .

**Proposition 6.12.** *Let  $\varphi : M \rightarrow \text{U}(n)$  be a nilconformal harmonic map from a surface which has image in  $\text{O}(n)$  or  $i\text{O}(n)$ , and let  $\alpha$  be an isotropic uniton for  $\varphi$  (possibly identically zero). Then*

- (i) *there is a real strict  $A_z^\varphi$ -filtration  $(Z_i)$  with  $Z_i = \alpha$  for some  $i$ .*
- (ii) *If  $\varphi$  maps into a real Grassmannian or into  $\text{O}(2m)/\text{U}(m)$ , and  $\alpha$  splits, then we can find such a filtration  $(Z_i)$  which splits.*

*Proof.* (i) Let  $\alpha = Z_0 \supset Z_1 \supset \dots \supset Z_s \supset Z_{s+1} = \underline{0}$  be a strict *partial*  $A_z^\varphi$ -filtration, i.e., a strict filtration satisfying (3.4); for example  $Z_i = (A_z^\varphi)^i(\alpha)$ . As in Lemma 6.2(i),  $\underline{\mathbb{C}}^n = \overline{Z}_{s+1}^\perp \supset \overline{Z}_s^\perp \supset \dots \supset \overline{Z}_1^\perp \supset \overline{Z}_0^\perp = \overline{\alpha}^\perp$  is also a strict partial  $A_z^\varphi$ -filtration. We can put them together to give a filtration

$$(6.4) \quad \underline{\mathbb{C}}^n = \overline{Z}_{s+1}^\perp \supset \overline{Z}_s^\perp \supset \dots \supset \overline{Z}_1^\perp \supset \overline{Z}_0^\perp = \overline{\alpha}^\perp \supset \alpha = Z_0 \supset Z_1 \supset \dots \supset Z_s \supset Z_{s+1} = \underline{0},$$

which is a real strict  $A_z^\varphi$ -filtration except that  $A_z^\varphi(\overline{\alpha}^\perp)$  may not lie in  $\alpha$ , or  $\overline{\alpha}^\perp$  may equal  $\alpha$ .

So let  $t = t(\alpha) \in \{-1, 0, 1, \dots\}$  be the least integer such that  $(A_z^\varphi)^{t+1}(\bar{\alpha}^\perp) \subset \alpha$ ; this exists by nilconformality.

(a) If  $t = -1$ , i.e.,  $\bar{\alpha}^\perp = \alpha$ , remove  $\bar{\alpha}^\perp$  from (6.4) leaving a real strict  $A_z^\varphi$ -filtration  $(Z_i)$  of length  $2s + 1$  with middle subbundle equal to  $\alpha$ .

(b) If  $t = 0$ , then  $A_z^\varphi(\bar{\alpha}^\perp) \subset \alpha$  and (6.4) is a real strict  $A_z^\varphi$ -filtration of length  $2s + 2$ .

(c) Otherwise, we have  $t \geq 1$ ; set  $\alpha_1 = (A_z^\varphi)^t(\bar{\alpha}^\perp) + \alpha$ . Then  $\alpha_1$  is a unitor which contains  $\alpha$ . Further,  $\alpha_1$  is isotropic, indeed, for the standard complex symmetric  $\mathbb{C}$ -bilinear inner product  $\langle \cdot, \cdot \rangle_{\mathbb{C}}$  on  $\mathbb{C}^n$ , since  $A_z^\varphi$  is symmetric,

$$\langle \alpha_1, \alpha_1 \rangle_{\mathbb{C}} = \langle (A_z^\varphi)^t(\bar{\alpha}^\perp), (A_z^\varphi)^t(\bar{\alpha}^\perp) \rangle_{\mathbb{C}} = \langle \bar{\alpha}^\perp, (A_z^\varphi)^{2t}(\bar{\alpha}^\perp) \rangle_{\mathbb{C}};$$

this is zero since  $2t \geq t + 1$ .

Thus we obtain a filtration:  $\bar{\alpha}^\perp \supset \bar{\alpha}_1^\perp \supset \alpha_1 \supset \alpha$  with  $A_z^\varphi(\alpha_1) \subset \alpha$  and  $A_z^\varphi(\bar{\alpha}_1^\perp) \subset \bar{\alpha}_1^\perp$ . Further  $(A_z^\varphi)^t(\bar{\alpha}_1^\perp) \subset (A_z^\varphi)^t(\bar{\alpha}^\perp) \subset \alpha_1$ , so that  $t(\alpha_1) \leq t(\alpha) - 1$ .

By repeating this construction at most  $t$  times we obtain a partial  $A_z^\varphi$ -filtration:

$$\bar{\alpha}^\perp \supset \bar{\alpha}_1^\perp \supset \bar{\alpha}_2^\perp \supset \dots \supset \bar{\alpha}_j^\perp \supset \alpha_j \supset \dots \supset \alpha_2 \supset \alpha_1 \supset \alpha.$$

Gluing this into the middle of (6.4) gives a real  $A_z^\varphi$ -filtration. If  $\bar{\alpha}_j^\perp \neq \alpha_j$ , this is a real strict  $A_z^\varphi$ -filtration of even length. If  $\bar{\alpha}_j^\perp = \alpha_j$ , remove  $\bar{\alpha}_j^\perp$ , leaving a real strict  $A_z^\varphi$ -filtration of odd length.

(ii) This is clear from the construction.  $\square$

**Example 6.13.** (i) If  $\alpha$  is a *basic* isotropic unitor, then it can be taken to be the last leg of the filtration, so that  $\bar{\alpha}$  is the first.

(ii) If  $t \in \{0, 1, \dots\}$  is the least integer such that  $(A_z^\varphi)^{t+1} = 0$ . Then, for any  $(t+1)/2 \leq s \leq t$ ,  $\text{Im}(A_z^\varphi)^s$  is an isotropic unitor, with the last one,  $\text{Im}(A_z^\varphi)^t$ , basic.

(iii) If  $n = 2m + 1$  is odd and  $\alpha$  is an isotropic unitor of rank  $m$ , then  $t(\alpha) = 1$ , i.e.,  $A_z^\varphi(\bar{\alpha}^\perp) \subset \alpha$  and we have case (b) above, so that we obtain a strict  $A_z^\varphi$ -filtration with  $\bar{\alpha}^\perp \supset \alpha$  in the middle. Indeed, if we had  $t(\alpha) > 1$ , then  $A_z^\varphi$  would factor to a non-zero map on the rank one bundle  $\bar{\alpha}^\perp/\alpha$  which is not possible by nilconformality.

Our next result concerns the Grassmanian  $\tilde{G}_k(\mathbb{R}^n)$  of oriented  $k$ -dimensional subspaces of  $\mathbb{R}^n$  discussed in §6.1. Recall that we have a double covering  $\tilde{G}_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n)$  which forgets the orientation: we call a smooth map  $\varphi : M \rightarrow \tilde{G}_k(\mathbb{R}^n)$  nilconformal if its composition with this double covering is nilconformal.

**Proposition 6.14.** *Let  $\varphi : M \rightarrow \tilde{G}_k(\mathbb{R}^n)$  be a harmonic map from a surface with  $k$  or  $n - k$  even. Then  $\varphi$  or  $\varphi^\perp$  has a  $J_2$ -holomorphic lift  $\psi : M \rightarrow F_{d_0, d_1, \dots, d_s}^{\mathbb{R}}$ , for some  $(d_0, d_1, \dots, d_s)$  satisfying (6.1), if and only if  $\varphi$  is nilconformal.*  $\square$

*Proof.* Suppose that  $\varphi$  or  $\varphi^\perp$  has a  $J_2$ -holomorphic lift as stated. Then  $\varphi$  is nilconformal by Corollary 3.12.

Conversely, suppose that  $\varphi$  is nilconformal. By replacing  $\varphi$  by  $\varphi^\perp$ , if necessary, we can assume that  $n - k$  is even.

(a) We find a maximal isotropic holomorphic subbundle  $W$  of  $\varphi^\perp$  which is closed under  $(A_z^\varphi)^2$ . This is done by a modification of the argument in the proof of Proposition 6.12, as follows.

Let  $\beta$  be an isotropic holomorphic subbundle of  $\varphi^\perp$  which is closed under  $(A_z^\varphi)^2$ . Note that this implies that  $\bar{\beta}^\perp$  is also closed under  $(A_z^\varphi)^2$ .

(i) We extend  $\beta$  to an isotropic holomorphic subbundle of  $\varphi^\perp$  which satisfies

$$(6.5) \quad (A_z^\varphi)^2(\bar{\beta}^\perp \cap \varphi^\perp) \subset \beta.$$

To do this, let  $u = u(\beta)$  be the least integer such that  $(A_z^\varphi)^{2u+2}(\bar{\beta}^\perp \cap \varphi^\perp) \subset \beta$ ; this exists by nilconformality. If  $u < 1$ ,  $\beta$  already satisfies (6.5). Otherwise  $u \geq 1$  and we set  $\beta_1 = (A_z^\varphi)^{2u}(\bar{\beta}^\perp \cap \varphi^\perp) + \beta$ . Then it is easy to check that  $\beta_1$  is isotropic, closed under  $(A_z^\varphi)^2$ ,  $\beta_1$  and has  $u(\beta_1) < u(\beta)$ . By repeating this construction at most  $u$  times starting with  $\beta = 0$  we obtain an isotropic holomorphic subbundle  $\beta$  of  $\varphi^\perp$  which satisfies (6.5).

(ii) We extend this  $\beta$  to a *maximal isotropic* holomorphic subbundle  $W$  of  $\varphi^\perp$  as in [7, Theorem 2.5] as follows. First note that the bundle  $\varphi^\perp \rightarrow M$  is oriented since it is the pull-back of the oriented tautological bundle  $L^\perp \rightarrow \tilde{G}_{n-k}(\mathbb{R}^n)$ . Then, starting with  $X = \beta$ , we can successively extend  $X$  increasing its rank by one until we obtain an isotropic holomorphic subbundle  $X$  with  $\text{rank } \overline{X}^\perp - \text{rank } X = 2$ . Then, by orientability of the bundle  $\varphi^\perp \rightarrow M$ , there is precisely one positive maximally isotropic subbundle  $W$  of  $\varphi^\perp$  containing  $X$ . Since  $\overline{\beta}^\perp \cap \varphi^\perp \supset W \supset \beta$ , by (6.5) we have  $(A_z^\varphi)^2(W) \subset W$ , as desired.

(b) Let  $W$  be a maximal isotropic holomorphic subbundle of  $\varphi^\perp$  which is closed under  $(A_z^\varphi)^2$ . Set  $\alpha = W + A_z^\varphi(W)$ . Then  $\alpha$  is an isotropic uniton. Note that  $\overline{\alpha}^\perp = W + A$  where  $A = \overline{A_z^\varphi(W)}^\perp \cap \varphi$  so that  $\overline{\alpha}^\perp \ominus \alpha \subset \varphi$ . From  $\langle A_z^\varphi(A), W \rangle_{\mathbb{C}} = \langle A_z^\varphi(W), A \rangle_{\mathbb{C}} = 0$ , we see that  $A_z^\varphi(A) \subset W$  whence  $A_z^\varphi(\overline{\alpha}^\perp) \subset \alpha$ . Let  $s$  be the least integer such that

$$(6.6) \quad (A_z^\varphi)^s(W) = 0, \quad \text{equivalently,} \quad (A_z^\varphi)^s(\alpha) = 0.$$

Set  $Z_i = (A_z^\varphi)^{i-s-1}(\alpha)$  ( $i = s+1, s+2, \dots, 2s+1$ ) and  $Z_i = \overline{Z}_{2s+1-i}^\perp$  ( $i = 0, 1, \dots, s$ ). Then  $(Z_i)$  is an alternating real  $A_z^\varphi$ -filtration of length  $2s$  with  $Z_s = \overline{\alpha}^\perp$ ,  $Z_{s+1} = \alpha$  and  $Z_{2s} = (A_z^\varphi)^{s-1}(W) \in \tilde{\varphi}$  where  $\tilde{\varphi} = (-1)^s \varphi$ . Setting  $\psi_i = Z_i \ominus Z_{i+1}$  defines a moving flag  $(\psi_0, \psi_1, \dots, \psi_{2s})$  which satisfies the  $J_2$ -holomorphicity condition (3.3). Now all  $\psi_i$  are non-zero with the possible exception of  $\psi_s = \overline{\alpha}^\perp \ominus \alpha$ . If this is zero, remove it and combine the legs  $\psi_{s-1}$  and  $\psi_{s+1}$  as in Operation 3 of Lemma 3.8, thus reducing  $s$  by one. Thus we obtain a  $J_2$ -holomorphic lift  $\psi : M \rightarrow F_{d_0, \dots, d_s}^{\mathbb{R}}$  of  $\tilde{\varphi}$  for some  $(d_0, \dots, d_s)$ .  $\square$

**Remark 6.15.** (i) Unlike Corollary 6.10, this corollary applies to nilconformal harmonic maps whether they have finite uniton number or not; however, it is not as explicit, as the proof, in general involves the choice of a holomorphic subbundle which is maximal isotropic.

(ii) Let  $\varphi : M \rightarrow G_k(\mathbb{R}^n)$  be a harmonic map from a surface to a real Grassmannian  $G_k(\mathbb{R}^n)$  of *unoriented* subspaces, with  $k$  or  $n-k$  even. Then  $\varphi$  or  $\varphi^\perp$  has a  $J_2$ -holomorphic lift  $\psi : M \rightarrow F_{d_0, d_1, \dots, d_s}^{\mathbb{R}}$ , for some  $(d_0, d_1, \dots, d_s)$  satisfying (6.1), if and only if  $\varphi$  is nilconformal with corresponding subbundle  $\varphi$  orientable, equivalently, the first Steifel–Whitney class  $w_1(L)$  of the tautological bundle  $L \rightarrow G_k(\mathbb{R}^n)$  satisfies  $\varphi^* w_1(L) = 0$ . Indeed, under those conditions  $\varphi$  lifts to a map into  $\tilde{G}_k(\mathbb{R}^n)$  and the theorem applies.

**Example 6.16.** (i) The condition (6.6) implies that

$$(6.7) \quad \text{Im}(A_z^\varphi)^s \cap \varphi^\perp \subset W \subset \ker(A_z^\varphi)^s \cap \varphi^\perp.$$

which in turn implies that  $(A_z^\varphi)^{2s}(\tilde{\varphi}^\perp) = 0$ . Conversely, if  $\varphi$  satisfies this last condition, then we can choose  $W$  to satisfy (6.7): just do the construction of part (i) of the proof above starting with  $\beta = \text{Im}(A_z^\varphi)^s \cap \varphi^\perp$ .

(ii) Putting  $s = 1$ , we deduce the following. Let  $\varphi : M \rightarrow \tilde{G}_k(\mathbb{R}^n)$  be a non-constant strongly conformal harmonic map with  $n-k$  even. Then (i) there are maximal isotropic holomorphic subbundles  $W$  of  $\varphi$  which satisfy (3.18); (ii) for such a  $W$ , we have a  $J_2$ -holomorphic twistor lift  $\psi = (\overline{W}, \varphi, W) : M \rightarrow F_{m,k}^{\mathbb{R}}$  of  $\varphi^\perp$ , where  $m = (n-k)/2$ , cf. Example 3.15(iii).

(iii) For  $n-k = 2$ , as in Example 3.15(iv), there is only one choice of  $W$  in (ii) and, reversing the roles of  $\varphi$  and  $\varphi^\perp$ , we obtain the following. Let  $\varphi : M \rightarrow G_2(\mathbb{R}^n)$  a non-constant harmonic map with  $\varphi^\perp$  strongly conformal. Then  $\varphi$  has a unique  $J_2$ -holomorphic lift  $\psi = (\overline{W}, \varphi^\perp, W) : M \rightarrow F_{1,n-2}^{\mathbb{R}}$ ; see also Corollary 7.4.

**6.4.  $J_2$ -holomorphic lifts for maps to the space of orthogonal complex structures.** The analogues of the results of §6.3 are as follows; the first following from the results of §6.3 and Corollary 6.23(iii) of [33].

**Proposition 6.17.** Let  $\Phi : M \rightarrow \Omega\text{U}(n)$  be a  $\nu$ -invariant polynomial extended solution which is real of some odd degree  $r = 2s+1$ . Then (i)  $n$  is even, i.e.,  $n = 2m$  for some  $m \in \mathbb{N}$ ; (ii)  $\varphi = \Phi_{-1} : M \rightarrow \text{O}(2m)/\text{U}(m)$  is a harmonic map of finite uniton number; (iii) all such harmonic maps  $\varphi$  are given this way up to sign; in fact, we may take  $\Phi$  to be normalized of degree at most  $2m-3$ .  $\square$



**Theorem 6.18.** *Let  $\Phi : M \rightarrow \Omega\mathrm{U}(2m)$  be a  $\nu$ -invariant polynomial extended solution which is normalized and real of odd degree  $r = 2s + 1$ . Let  $\varphi = \Phi_{-1} : M \rightarrow \mathrm{O}(2m)/\mathrm{U}(m)$  be the resulting harmonic map. Then  $\varphi$  has a  $J_2$ -holomorphic lift  $\psi : M \rightarrow \mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}}$  for some  $(d_0, d_1, \dots, d_s)$  with  $s \leq 2m - 3$  and  $\sum_{i=0}^s d_i = m$ , namely the canonical twistor lift defined by  $\Phi$  (see Theorem 4.8).  $\square$*

**Corollary 6.19.** *Let  $\varphi : M \rightarrow \mathrm{O}(2m)/\mathrm{U}(m)$  be a harmonic map of finite uniton number. Then either  $\varphi$  or  $-\varphi$  has a  $J_2$ -holomorphic twistor lift  $\psi : M \rightarrow \mathcal{Z}_{d_0, d_1, \dots, d_s}^{\mathbb{R}}$  for some  $(d_0, d_1, \dots, d_s)$  with  $s \leq 2m - 3$  and  $\sum_{i=0}^s d_i = m$ , namely the canonical twistor lift defined by a normalized extended solution  $\Phi$  with  $\Phi_{-1} = \pm\varphi$ .  $\square$*

**Example 6.20.** As in Example 6.11, let  $\underline{\varrho} = \beta_0 \subset \beta_1 \subset \dots \subset \beta_r \subset \beta_{r+1} = \underline{\mathbb{C}}^n$  be a real superhorizontal sequence. Set  $\psi_i = \beta_{i+1} \ominus \beta_i$ . If  $r$  is odd, say  $r = 2s + 1$ , then  $\varphi = \Phi_{-1} = \sum_j \psi_{2j}$  is a map into  $\mathrm{O}(2m)/\mathrm{U}(m)$ . The map  $\psi = (\psi_0, \psi_1, \dots, \psi_r) : M \rightarrow \mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}}$  is thus the canonical twistor lift of  $\varphi$  defined by  $\Phi$ ; again, as in §5.3, it is superhorizontal.

The construction of real superhorizontal sequences is discussed in [33, §6.4].

As for maps into real Grassmannians, we can actually find lifts for harmonic maps which are not of finite uniton number provided they are nilconformal, as follows.

**Proposition 6.21.** *Let  $\varphi : M \rightarrow \mathrm{O}(2m)/\mathrm{U}(m)$  be a harmonic map. Then  $\varphi$  or  $-\varphi$  has a  $J_2$ -holomorphic twistor lift  $\psi : M \rightarrow \mathcal{Z}_{d_0, \dots, d_s}^{\mathbb{R}}$  for some  $s$  and  $d_i$  if and only if  $\varphi$  is nilconformal.*

*Proof.* As before, if there is a twistor lift, then there is an  $A_x^{\varphi}$ -filtration so that  $\varphi$  is nilconformal.

Conversely, as in Proposition 6.12 we can construct a real  $A_x^{\varphi}$ -filtration which splits. Lemma 3.7(i) or (ii) then gives a moving flag which satisfies the  $J_2$ -holomorphicity condition; reality of the filtration implies that this flag is real. We can then apply Operations 1 and 2, and Operation 3 *symmetrically* (i.e., if a zero leg  $\psi_i$  is removed, so is its conjugate  $\psi_{s-i}$ ), to remove zero legs whilst preserving reality; once that is done, we are left with a twistor lift  $\psi$  as stated.  $\square$

## 7. HARMONIC MAPS INTO QUATERNIONIC SPACES

**7.1. Twistor lifts of maps into quaternionic Grassmannians and  $\mathrm{Sp}(m)/\mathrm{U}(m)$ .** The results of the previous section for the orthogonal group hold for the symplectic group  $\mathrm{Sp}(m)$ , with a few modifications. We give here some definitions, and refer to [25] and [33, §6.8] for more results on harmonic maps into  $\mathrm{Sp}(m)$ .

To define the relevant twistor spaces, let  $J$  be the conjugate linear endomorphism of  $\mathbb{C}^{2m} \cong \mathbb{H}^m$  corresponding to left multiplication by the quaternion  $j$ . Let  $d_0, d_1, \dots, d_s$  be positive integers with  $d_s + 2 \sum_{i=0}^{s-1} d_i = m$ , and set  $d_i = d_{2s-i}$  for  $i = s+1, \dots, 2s$ . Define a submanifold  $F_{d_0, \dots, d_{2s}}^J \subset F_{d_0, \dots, d_{2s}}$  by

$$F_{d_0, \dots, d_{2s}}^J = \{ \psi = (\psi_0, \psi_1, \dots, \psi_{2s}) \in F_{d_0, \dots, d_{2s}} : \psi_i = J\psi_{2s-i} \ \forall i \}.$$

Note that the middle leg  $\psi_s$  is *quaternionic*, i.e.,  $J\psi_s = \psi_s$ .

Similarly, let  $d_0, d_1, \dots, d_s$  be positive integers with  $d_0 + \dots + d_s = m$ , set  $d_i = d_{2s+1-i}$  for  $i = s, \dots, 2s+1$ , and define a submanifold  $\mathcal{Z}_{d_0, \dots, d_s}^J \subset F_{d_0, \dots, d_{2s+1}}$  by

$$\mathcal{Z}_{d_0, \dots, d_s}^J = \{ \psi = (\psi_0, \psi_1, \dots, \psi_{2s+1}) \in F_{d_0, \dots, d_{2s+1}} : \psi_i = J\psi_{2s+1-i} \ \forall i \}.$$

As in the previous section, the projection (3.2) restricts to homogeneous projections  $\pi_e^J$  from  $F_{d_0, \dots, d_s}^J$  to the quaternionic Grassmannian  $G_k(\mathbb{H}^m) = \mathrm{Sp}(m)/\mathrm{Sp}(k) \times \mathrm{Sp}(m-k)$  where  $k = \sum_{i=0}^s d_{2i}$ , and from  $\mathcal{Z}_{d_0, \dots, d_s}^J$  to the space  $\mathrm{Sp}(m)/\mathrm{U}(m)$  of quaternionic complex structures on  $\mathbb{C}^{2m}$ .

A map  $\varphi : M \rightarrow \mathrm{U}(2m)$  takes values in the subgroup  $\mathrm{Sp}(m)$  if and only if  $J\varphi = \varphi J$ . Let  $r$  be an integer; then an extended solution  $\Phi$  is said to be *symplectic (of degree  $r$ )* if  $J\Phi J^{-1} = \lambda^{-r}\Phi$ . Set  $W = \Phi\mathcal{H}_+$ ; then  $\Phi$  is symplectic (of degree  $r$ ) if and only if  $JW^\perp = \lambda^{1-r}W$ , in which case  $W$  is also said to be symplectic (of degree  $r$ ). On setting  $\varphi = \Phi_{-1}$ , it follows that  $\varphi$  (if  $r$  is even) or  $i\varphi$  (if  $r$  is odd) takes values in  $\mathrm{Sp}(m)$ . If  $\Phi$  is  $\nu$ -invariant, then  $\varphi$  takes values in an inner symmetric

space of  $\mathrm{Sp}(m)$ , more specifically, a quaternionic Grassmannian (if  $r$  is even) or  $\mathrm{Sp}(m)/\mathrm{U}(m)$  (if  $r$  is odd).

Given a polynomial,  $\nu$ -invariant, symplectic extended solution  $\Phi$ , we obtain from the canonical filtration of  $W = \Phi\mathcal{H}_+$  a twistor lift  $\psi$  of  $\varphi = \Phi_{-1}$  with values in either  $F_{d_0, \dots, d_s}^J$  or  $\mathcal{Z}_{d_0, \dots, d_s}^J$  according as  $r = 2s$  or  $r = 2s + 1$ . This is proved in the same way as was done for the orthogonal group in the previous section.

We obtain similar theorems to those of Sections 6.3 and 6.4; we leave the reader to write these down.

**Example 7.1.** A superhorizontal sequence  $\underline{\mathcal{Q}} = \beta_0 \subset \beta_1 \subset \dots \subset \beta_r \subset \beta_{r+1} = \mathbb{C}^{2m}$  is said to be *symplectic* if  $J\beta_i^\perp = \beta_{r-i}$  for all  $i$ . Writing, as before,  $\psi_i = \beta_{i+1} \ominus \beta_i$  and  $d_i = \mathrm{rank} \psi_i$ , then  $\psi = (\psi_0, \psi_1, \dots, \psi_r)$  satisfies the superhorizontality condition (3.3); so if the  $d_i$  are all non-zero,  $\psi$  is a superhorizontal holomorphic map with values in  $F_{d_0, \dots, d_s}^J$  (if  $r = 2s$ ) or  $\mathcal{Z}_{d_0, \dots, d_s}^J$  (if  $r = 2s + 1$ ).

Set  $\Phi = \sum_{i=0}^r \lambda^i \pi_{\psi_i}$ ; equivalently,  $W = \Phi\mathcal{H}_+ = \sum_{i=0}^{r-1} \lambda^i \beta_i + \lambda^r \underline{\mathcal{H}}_+$ . Then  $\Phi$  is an  $S^1$ -invariant extended solution, symplectic of degree  $r$ , and  $\Phi_{-1} = \pi_e^J \circ \psi$ . Conversely, any symplectic  $S^1$ -invariant extended solution is given this way.

Recall that a full holomorphic map  $h : M \rightarrow \mathbb{C}P^{2m-1}$  is said to be *totally  $J$ -isotropic* if  $G^{(2m-1)}(h) = Jh$  [2]. Then setting  $\beta_i = h_{(i-1)}$  defines a superhorizontal symplectic sequence of length  $2m - 1$ ; the superhorizontal holomorphic map  $\psi$  takes values in  $\mathcal{Z}_{1, \dots, 1}^J$  (with  $m$  1s) and is a  $J_2$ -holomorphic twistor lift of a harmonic map into  $\mathrm{Sp}(m)/\mathrm{U}(m)$ .

**Example 7.2.** (i) Recall from Example 5.11 that a  $\partial'$ -pair (i.e., superhorizontal sequence of length 2)  $\underline{\mathcal{Q}} \subset \beta_1 \subset \beta_2 \subset \mathbb{C}^n$  gives rise to two harmonic maps: the mixed pair  $\varphi = \beta_1 \oplus \beta_2^\perp$  and the strongly isotropic map  $\varphi^\perp = \beta_2 \ominus \beta_1$ . Suppose that  $n = 2m$ . Then the harmonic maps  $\varphi$  and  $\varphi^\perp$  are quaternionic, i.e., have image in a quaternionic Grassmannian  $G_*(\mathbb{H}^m)$ , if and only if  $\beta_2^\perp = J\beta_1$ , in which case  $(\beta_1, \beta_2^\perp) = (\beta_1, J\beta_1)$  is called a *quaternionic mixed pair* [2] and  $\varphi = \beta_1 \oplus J\beta_1$ . In this case, the canonical lift of  $\varphi$  defined by the extended solution (5.11) is the superhorizontal holomorphic map  $\psi = (\beta_1, \varphi^\perp, J\beta_1) : M \rightarrow F_{d_0, d_1}^{\mathbb{R}}$  where  $d_0 = \mathrm{rank} \beta_1$ ,  $d_1 = 2m - 2d_0$ .

(ii) In the case that  $\beta_1$  has rank one, we have  $\varphi = \beta_1 \oplus J\beta_1 : M \rightarrow \mathbb{H}P^{m-1}$ . We may identify  $F_{1, 2m-2}^J$  with  $\mathbb{C}P^{2m-1}$  via the map  $(\psi_0, \psi_1, \psi_2) \mapsto \psi_0$ . With this identification,  $\pi_e^{\mathbb{R}} : \mathbb{C}P^{2m-1} \rightarrow \mathbb{H}P^{m-1}$  is the standard Riemannian fibration which maps  $L \in \mathbb{C}P^{2m-1}$  to  $L \oplus JL \in \mathbb{H}P^{m-1}$ , and the canonical lift of  $\varphi$  gives the superhorizontal holomorphic map  $\beta_1 \cong (\beta_1, \varphi^\perp, J\beta_1) : M \rightarrow \mathbb{C}P^{2m-1} \cong F_{1, 2m-2}^J$ .

(iii) Let  $h : M \rightarrow \mathbb{C}P^{2m-1}$  be a full totally  $J$ -isotropic map. Then  $JG^{(m-1)}(h) = G^{(m)}(h)$ , and the harmonic map  $\varphi^\perp = G^{(m-1)}(h) \oplus G^{(m)}(h) : M \rightarrow \mathbb{H}P^{m-1}$  is called a *quaternionic Frenet pair* [2]. As in Example 5.11(iii), set  $\beta_1 = h_{(m-2)}$  and  $\beta_2 = h_{(m)}$ . Then  $J\beta_2^\perp = \beta_1$  and the canonical lift of  $\varphi$  defined by the extended solution (5.11) is the superhorizontal holomorphic map  $(\beta_1, \varphi^\perp, J\beta_1) : M \rightarrow F_{m-1, 2}^J$ . Since  $\varphi$  is strongly conformal,  $\varphi^\perp$  also has a (unique) twistor lift as in Example 3.15(iv), namely the  $J_2$ -holomorphic map  $G^{(m)}(h) \cong (G^{(m)}(h), \varphi, G^{(m-1)}(h)) = (G''(\varphi), \varphi, G'(\varphi)) : M \rightarrow \mathbb{C}P^{2m-1} \cong F_{1, 2m-2}^J$ .

In contrast to harmonic maps from the 2-sphere to real and complex projective spaces, harmonic maps from the 2-sphere to quaternionic projective spaces are harder to describe: see [2] for a method of reduction to Frenet and mixed pairs, and see [25, 33] for unitor factorizations; however, there is one important class that we can completely describe, we turn to that class now.

**7.2. Inclusive harmonic maps into quaternionic Kähler manifolds.** Recall [30] that a *quaternionic Kähler manifold*  $N^{4n}$  is a real oriented  $4n$ -dimensional Riemannian manifold whose holonomy belongs to the subgroup  $\mathrm{Sp}(n)\mathrm{Sp}(1)$  of  $\mathrm{SO}(4n)$ . Such a manifold has a natural  $\mathbb{C}P^1$ -bundle  $Q \rightarrow N$  whose fibre at a point  $q \in N$  consists of all the orthogonal complex structures on  $T_q N$  which are ‘compatible’ with the  $\mathrm{Sp}(n)\mathrm{Sp}(1)$  structure; we will call  $Q$  the *twistor space of  $N$  in the quaternionic sense*. Any oriented Riemannian 4-manifold satisfies the above definition with  $Q \rightarrow N$  equal to the bundle  $Z^+ \rightarrow N$  of positive almost Hermitian structures, but, as this dimension is exceptional, most authors insist that  $n \geq 2$  in the definition of ‘quaternionic Kähler’.

We call a subspace of  $TN$  *quaternionic* if it is closed under  $Q$ . A weakly conformal map  $\varphi : M \rightarrow N$  from a Riemann surface to a quaternionic Kähler manifold is called *inclusive* [31, 15]

if, for each  $p \in M$ ,  $d\varphi(T_p M)$  is contained in a 4-real-dimensional quaternionic subspace  $S_p$  of  $T_{\varphi(p)} N$ . This is equivalent to saying that, for each  $p \in M$ , there is a  $q_p \in Q_{\varphi(p)}$  with respect to which  $\varphi$  is holomorphic, i.e., its differential intertwines the complex structure on  $T_p M$  with  $q_p$ .

If  $d\varphi(p)$  is non-zero,  $d\varphi(\partial/\partial z)$  spans an isotropic subspace;  $S_p$  and  $q_p$  are determined uniquely by that. If  $\varphi$  is harmonic, then  $d\varphi(\partial/\partial z)$  is holomorphic with respect to the Koszul–Malgrange structure on  $\varphi^{-1}T^c N$  (see, for example, [7, Chapter 2]); as usual, we can fill out zeros to extend the span of  $d\varphi(\partial/\partial z)$ , and so  $S$  and  $q$ , smoothly across the zeros of  $d\varphi$ . Thus a (weakly conformal) inclusive harmonic map has a twistor lift  $\psi : M \rightarrow Q$ ; Eells and Salamon [15] showed that this lift is  $J_2$ -holomorphic, establishing that *there is a one-to-one correspondence between inclusive weakly conformal harmonic maps  $\varphi : M \rightarrow N$  and  $J_2$ -holomorphic maps  $\psi : M \rightarrow Q$  which project to  $\varphi$* . We identify this correspondence for the three quaternionic Kähler manifolds, the Grassmannians  $\tilde{G}_4(\mathbb{R}^n)$ ,  $G_2(\mathbb{C}^n)$  and quaternionic projective space  $\mathbb{H}P^{m-1}$ .

(i) First, we consider the real Grassmannian  $N = \tilde{G}_4(\mathbb{R}^n) = \text{SO}(n)/\text{SO}(4) \times \text{SO}(n-4)$  of oriented 4-dimensional subspaces of  $\mathbb{R}^n$ . Taking the orthogonal complement of a subspace identifies this with  $\tilde{G}_{n-4}(\mathbb{R}^n)$ ; then, for each  $Y \in N$ ,  $T_Y N$  can be identified with the space  $\text{Hom}_{\mathbb{R}}(Y^\perp, Y)$  of real linear maps. Set  $Q_Y^+$  (resp.  $Q_Y^-$ ) equal to the set of almost complex structures on  $T_Y N$  given by postcomposition of an element of  $\text{Hom}(Y^\perp, Y)$  with a positive (resp. negative) almost Hermitian structure on  $Y$ . Then the bundle  $Q^+ \rightarrow N$  is the twistor space of  $N$  in the quaternionic sense; to see  $Q^- \rightarrow N$  as a quaternionic Kähler structure, we must put the other orientation on  $N$  or proceed as follows.

We may identify  $Q_Y^+$  (resp.  $Q_Y^-$ ) with the space of maximal positive (resp. negative) isotropic subspaces of  $Y$  by associating to  $q \in Q$  its  $(0, 1)$ -space  $V$ ; thus the bundle  $Q^+ \rightarrow N$  can be identified with  $\pi_e^{\mathbb{R}} : F_{2,n-4}^{\mathbb{R}} \rightarrow N$ . Let  $A : \tilde{G}_{n-4}(\mathbb{R}^n) \rightarrow \tilde{G}_{n-4}(\mathbb{R}^n)$  be the map which sends each subspace to the same subspace with the opposite orientation: thus for  $n = 5$ ,  $A : \tilde{G}_1(\mathbb{R}^5) = S^4 \rightarrow S^4$  is the antipodal map  $A(x) = -x$ ; then  $Q^- \rightarrow N$  can be identified with the bundle  $A \circ \pi_e^{\mathbb{R}} : F_{2,n-4}^{\mathbb{R}} \rightarrow N$ .

Let  $i : \tilde{G}_{n-4}(\mathbb{R}^n) \rightarrow G_{n-4}(\mathbb{R}^n) \hookrightarrow G_{n-4}(\mathbb{C}^n)$  be the canonical immersion (see §6.1). We have the following result.

**Proposition 7.3.** *Let  $\varphi : M \rightarrow \tilde{G}_4(\mathbb{R}^n)$  be a non-constant weakly conformal harmonic. Then either  $\varphi$  or  $A \circ \varphi$  is inclusive if and only if  $i \circ \varphi^\perp$  is strongly conformal. In this case  $\varphi$  or  $A \circ \varphi$  has a  $J_2$ -holomorphic lift  $\psi : M \rightarrow Q^\pm = F_{2,n-4}^{\mathbb{R}}$ .*

*Proof.* By definition,  $i \circ \varphi^\perp$  strongly conformal means that, at each point  $p \in M$ , the Gauss transforms  $G'(i \circ \varphi^\perp)$  and  $G''(i \circ \varphi^\perp)$  are orthogonal. Now, under the inclusion map  $i$ ,  $d\varphi_p(\partial/\partial z)$  and  $d\varphi_p(\partial/\partial \bar{z})$  map to  $A'_{i \circ \varphi^\perp}$  and  $A''_{i \circ \varphi^\perp}$ , respectively, so that strong conformality of  $\Phi$  is equivalent to the image of  $d\varphi_p(\partial/\partial z)$  being an isotropic subspace of  $\varphi(p) \times \mathbb{C}$  of dimension one or two. By filling out zeros we obtain the isotropic image subbundle  $\text{Im } d\varphi(\partial/\partial z)$  of rank one or two.

If  $\text{Im } d\varphi(\partial/\partial z)$  is of rank one, then there are precisely two isotropic subbundles of  $\varphi(p) \times \mathbb{C}$  of rank two containing it, giving two almost Hermitian structures  $q$  at each point, one positive and one negative. The positive one gives a lift of  $\varphi$  into  $Q^+$ ; the negative one gives a lift into  $Q^-$ , equivalently, of  $A \circ \varphi$  into  $Q^+$ .

If  $\text{Im } d\varphi(\partial/\partial z)$  is of rank two, then it defines a positive or negative almost Hermitian structure at each point, giving a lift into either  $Q^+$  or  $Q^-$ .  $\square$

(ii) We next consider the complex Grassmannian  $G_2(\mathbb{C}^n)$ . On identifying  $\mathbb{C}^n$  with  $\mathbb{R}^{2n}$ ,  $G_2(\mathbb{C}^n)$  can be considered as the totally geodesic submanifold of  $G_4(\mathbb{R}^{2n})$  given by  $\{Y \in G_4(\mathbb{R}^{2n}) : Y \text{ is complex}\}$ . We give  $G_2(\mathbb{C}^n)$  the conjugate of its canonical complex structure, i.e., that inherited from  $G_{n-2}(\mathbb{C}^n)$  by the identification  $G_2(\mathbb{C}^n) \rightarrow G_{n-2}(\mathbb{C}^n)$  given by  $Y \mapsto Y^\perp$ . Then the complexified tangent space to  $G_2(\mathbb{C}^n)$  at any  $Y$  can be identified with  $\text{Hom}_{\mathbb{C}}(Y^\perp, Y) \oplus \text{Hom}_{\mathbb{C}}(Y, Y^\perp)$ , where the summands are the  $(1, 0)$ - and  $(0, 1)$ -tangent spaces for this complex structure on  $G_2(\mathbb{C}^n)$ . The manifold  $G_2(\mathbb{C}^n)$  has a quaternionic Kähler structure with quaternionic twistor space  $\pi_e : F_{1,n-2,1} \rightarrow G_2(\mathbb{C}^n)$ . The almost Hermitian structure  $q_Y$  on  $T_Y G_2(\mathbb{C}^n)$  corresponding to  $(V, Y^\perp, W) \in F_{1,n-2,1}$  is that determined by the subspace  $V$  of  $Y$  (or its orthogonal complement  $W$  in  $Y$ ), explicitly, the  $(1, 0)$ -space of  $q_Y$  is  $\text{Hom}_{\mathbb{C}}(Y^\perp, V) \oplus \text{Hom}_{\mathbb{C}}(W, Y^\perp)$  (cf. [29, p. 125(iii)]).

We have an embedding of  $F_{1,n-2,1}$  in  $F_{2,2n-4}^{\mathbb{R}}$ , covering the inclusion of  $G_2(\mathbb{C}^n)$  in  $G_4(\mathbb{R}^{2n})$ , given by  $(V, Y^\perp, W) \mapsto (V + \overline{W}, Y^\perp, \overline{V} + W)$ .

Now, a map  $\varphi^\perp : M \rightarrow G_{n-2}(\mathbb{C}^n)$  is strongly conformal if and only if its composition with the inclusion mappings  $G_{n-2}(\mathbb{C}^n) \hookrightarrow G_{2n-4}(\mathbb{R}^{2n}) \hookrightarrow G_{2n-4}(\mathbb{C}^{2n})$  is strongly conformal, so we obtain the following: the first part is due to Rawnsley [9, §5C]; uniqueness comes from Example 3.15.

**Corollary 7.4.** *Let  $\varphi : M \rightarrow G_2(\mathbb{C}^n)$  be non-constant and weakly conformal. Then  $\varphi$  is inclusive if and only if  $\varphi^\perp$  is strongly conformal. In that case  $\varphi$  has a unique  $J_2$ -holomorphic lift  $\psi : M \rightarrow Q = F_{1,n-2,1}$  given by*

$$\psi = (G''(\varphi^\perp), \varphi^\perp, G'(\varphi^\perp)).$$

*On including  $G_2(\mathbb{C}^n)$  in  $G_4(\mathbb{R}^{2n})$  and  $F_{1,n-2,1}$  in  $F_{2,2n-4}^{\mathbb{R}}$ , this coincides with the lift given by Proposition 7.3.*

(iii) Lastly, we consider the quaternionic projective space  $\mathbb{H}P^{m-1}$ . On identifying  $\mathbb{H}^m$  with  $\mathbb{C}^{2m}$ , the space  $\mathbb{H}P^{m-1}$  can be thought of as the totally geodesic submanifold of  $G_2(\mathbb{C}^{2m})$  given by  $\{Y \in G_2(\mathbb{C}^{2m}) : Y \text{ is quaternionic}\}$ . The quaternionic Kähler structure on  $G_2(\mathbb{C}^{2m})$  has quaternionic twistor space  $Q = \pi_e^J : F_{1,2m-2}^J \rightarrow \mathbb{H}P^{m-1}$ ; as in Example 7.2(ii), this is the standard fibration  $\mathbb{C}P^{2m-1} \rightarrow \mathbb{H}P^{m-1}$ . We have an embedding of  $F_{1,2m-2}^J$  in  $F_{1,2m-2,1}$ , covering the inclusion of  $\mathbb{H}P^{m-1}$  in  $G_2(\mathbb{C}^{2m})$ , given by  $F_{1,2m-2}^J \cong \mathbb{C}P^{2m-1} \ni V \mapsto (V, (V + JV)^\perp, JV) \in F_{1,2m-2,1}$ .

Using  $G''(\varphi) = JG'(\varphi)$ , we deduce the following from Corollary 7.4, cf. [9, Proposition 5.7].

**Corollary 7.5.** *Let  $\varphi : M \rightarrow \mathbb{H}P^{m-1}$  be non-constant and weakly conformal. Then it is inclusive if and only if  $\varphi$  is reducible, i.e., its Gauss transform  $G'(\varphi)$  has rank one. In this case,  $G'(\varphi^\perp)$  also has rank one and  $\varphi$  has a unique  $J_2$ -holomorphic lift into  $F_{1,2m-2}^J$  given by  $\psi = (G'(\varphi^\perp), \varphi^\perp, JG'(\varphi^\perp))$ . On including  $F_{1,2m-2}^J$  in  $F_{1,2m-2,1}$ , this agrees with the lift given in Corollary 7.4.*

## 8. EXPLICIT FORMULAE FOR TWISTOR LIFTS

**8.1. Formulae from the Grassmannian model.** Corollary 5.3 describes how twistor lifts which are  $J_2$ -holomorphic can be obtained from (partial) unitor factorizations with basic unitons. We show how to find explicit formulae for these lifts.

First of all, we recall from [33] how to find explicit formulae for all polynomial extended solutions, and thus for all harmonic maps of finite unitor number from a surface into  $U(n)$ , in terms of arbitrary holomorphic data.

We need the following construction of M. A. Guest [22]: Let  $r \in \mathbb{N}$  and let  $X$  be an arbitrary holomorphic subbundle of  $(\mathbb{C}^r, \partial_{\bar{z}})$ . Set  $W$  equal to the following subbundle of  $\mathcal{H}_+$ :

$$(8.1) \quad W = X + \lambda X_{(1)} + \lambda^2 X_{(2)} + \cdots + \lambda^{r-1} X_{(r-1)} + \lambda^r \mathcal{H}_+,$$

where  $X_{(i)}$  denotes the  $i$ th osculating space of  $X$  as in Example 5.6. Then  $W$  is an extended solution satisfying (4.2), and all such  $W$  are given this way (we can take  $X = W$ ); we shall say that  $X$  *generates*  $W$ . We shall describe subbundles of a trivial holomorphic bundle  $(\mathbb{C}^N, \partial_{\bar{z}})$  ( $N \in \mathbb{N}$ ) by giving *meromorphic spanning sets* for them, as in [33, §4.1]. As in that paper, by the *order*  $o(L)$  of a meromorphic section  $L$  of  $\mathcal{H}_+$  we mean the least integer  $i$  such that  $P_i L \neq 0$ ; equivalently,  $L = \lambda^{o(L)} \widehat{L}$  for some  $\widehat{L} = \sum_{\ell \geq 0} \lambda^\ell \widehat{L}_\ell$  with  $\widehat{L}_0$  non-zero. Let  $\{L_j\}$  be a meromorphic spanning set for  $X$ . Then a meromorphic spanning set for  $W \bmod \lambda W$  is  $\{\lambda^k (L_j)^{(k)} : 0 \leq k \leq r\}$ .

Now choose a unitor or partial unitor factorization (5.3) of  $W$  satisfying (5.6). Thus, writing  $W = \Phi \mathcal{H}_+$ , the extended solution  $\Phi$  has (partial) unitor factorization (5.1) with unitons  $\alpha_i$  which are basic for  $\Phi_{i-1}$ . The corresponding  $F$ -filtration  $(Y_i)$  is given by (5.5) and the  $A_z^\mathcal{E}$ -filtration corresponding to that is given by  $Z_i = P_0 \circ \Phi^{-1} Y_i$ . If  $\{H_j^i\}$  is a meromorphic spanning set for  $Y_i \bmod \lambda W$ , then  $\{P_0 \circ \Phi^{-1} H_j^i\}$  is a meromorphic spanning set for  $Z_i$ ; then a basis for the legs  $\psi_i = Z_i \ominus Z_{i+1}$  can be found from that set by the Gram–Schmidt process. If all the legs are non-zero, this gives the  $J_2$ -holomorphic twistor lift of  $\varphi = \Phi_{-1}$  described in Corollary 5.3.

To calculate this explicitly, first, we need to find a meromorphic spanning set  $\{H_j^i\}$  for each  $Y_i$  from a meromorphic spanning set for  $W$ ; in the examples below, this is done by finding a

meromorphic spanning set for  $W$  adapted to the filtration  $(Y_i)$ . Second, let  $S_s^r$  denote the sum of all  $r$ -fold products of the form  $\Pi_r \cdots \Pi_1$  where exactly  $s$  of the  $\Pi_j$  are  $\pi_{\alpha_j}^\perp$  and the other  $r - s$  are  $\pi_{\alpha_j}$ . Then given a meromorphic spanning set  $\{H_j^i\}$  for  $Y_i \bmod \lambda W$ , the meromorphic spanning set  $\{P_0 \circ \Phi^{-1} H_j^i\}$  for  $Z_i$  is given by  $P_0 \circ \Phi^{-1} H_j^i = \sum_{s=k}^r S_s^r P_s(H_j^i)$ . We now see how this works for our two main examples; other examples can be done similarly.

**Example 8.1.** We find explicit formulae for the canonical twistor lift of the harmonic map  $\varphi = \Phi_{-1}$  defined by a normalized extended solution  $W = \Phi \mathcal{H}_+$ . Let  $X$  generate  $W$  and let  $\{L_j\}$  be a meromorphic spanning set for  $X$ . The filtration  $(Y_i)$  which gives the canonical twistor lift is given by (4.7), and a meromorphic spanning set for  $Y_i \bmod \lambda W$  is  $\{\lambda^k (L_j)^{(k)} : i \leq o(L_j) + k \leq r\}$ .

Let  $\alpha_1, \dots, \alpha_r$  be the Uhlenbeck unitons of  $\Phi$ , see Example 5.5; then a meromorphic spanning set for the corresponding  $A_z^\varphi$ -filtration  $Z_i = P_0 \circ \Phi^{-1} Y_i$  is given by

$$Z_i = \text{span} \left\{ \sum_{s=k}^r S_s^r P_{s-k}(L_j)^{(k)} : i \leq o(L_j) + k \leq r \right\}.$$

Applying the Gram–Schmidt process gives explicit formulae for the canonical twistor lift  $\psi$  of  $\varphi = \Phi_{-1}$  defined by  $\Phi$ .

**Example 8.2.** Suppose that  $\varphi : M \rightarrow G_*(\mathbb{C}^n)$  is a nilconformal harmonic map. Let  $U \subset M$  be a domain on which it admits an associated extended solution; we find explicit formulae for Burstall’s twistor lift (Example 3.13) of  $\varphi$  on that domain. As in Lemma 4.3 we can find a  $\nu$ -invariant extended solution  $\Phi$  with  $\Phi_{-1} = \varphi$ . Set  $W = \Phi \mathcal{H}_+$  as usual. Let  $X$  generate  $W$  and let  $\{L_j\}$  be a meromorphic spanning set for  $X$ . The relevant filtration  $(Y_i)$  is described in Example 4.10; a meromorphic spanning set for  $Y_i \bmod \lambda Y_i$  is given by  $\{\lambda^k (L_j)^{(k)} : i \leq k \leq r\}$ . Hence a meromorphic spanning set for the corresponding  $A_z^\varphi$ -filtration  $Z_i = P_0 \circ \Phi^{-1} Y_i$  is given by

$$Z_i = \text{span} \left\{ \sum_{s=k}^r S_s^r P_{s-k}(L_j)^{(k)} : i \leq k \leq r \right\}.$$

Applying the Gram–Schmidt process gives explicit formulae for the twistor lift  $\psi$  of  $\pm\varphi$  defined in Example 3.13.

Note that, (i) in both cases, the harmonic map  $\varphi$  is given (a) as the product of unitons by the formulae in [33, §4.1], (b) as the sum  $\sum \psi_{2j}$  of the even-numbered legs of  $\psi$ ; (ii) we can generate *all* harmonic maps  $M \rightarrow G_*(\mathbb{C}^n)$  of finite uniton number, and the above twistor lifts of them, by *freely* choosing meromorphic functions  $L_j : M \rightarrow \mathbb{C}^n$  and computing the lifts as above, giving *completely explicit formulae for all harmonic maps of finite uniton number from a surface to a complex Grassmannian and their twistor lifts*.

## 8.2. Examples.

**Example 8.3.** Let  $\varphi$  be a harmonic map from a Riemann surface to  $G_2(\mathbb{C}^4)$  of (minimal) uniton number 3. We shall find a  $J_2$ -holomorphic twistor lift of  $\pm\varphi$ . By [33, Corollary 5.7], there is a polynomial extended solution  $\Phi$  of degree 3 with  $\Phi_{-1} = \pm\varphi$ . On replacing  $\varphi$  by its orthogonal complement if necessary, we may assume that  $\Phi_{-1} = \varphi$ . Set  $W = \Phi \mathcal{H}_+$ . This is closed under  $\nu$  and so has the form

$$(8.2) \quad W = \text{span}\{H_0 + \lambda^2 H_2\} + \lambda \delta_2 + \lambda^2 \delta_3 + \lambda^3 \underline{\mathcal{H}}_+$$

where  $H_0, H_2 : M \rightarrow \mathbb{C}^4$  are meromorphic maps (equivalently, meromorphic sections of the trivial bundle  $\underline{\mathbb{C}}^4$ ), the  $\delta_i$  are subbundles of  $\underline{\mathbb{C}}^4$ , and setting  $\delta_1 = \text{span}\{H_0\}$ , we have  $(\delta_1)_{(1)} \subset \delta_2$  and  $(\delta_2)_{(1)} \subset \delta_3$ .

Now, none of the  $\delta_i$  is constant, otherwise,  $(\pi_{\delta_i} + \lambda^{-1} \pi_{\delta_i})\Phi$  would be a polynomial extended solution of degree 2 associated to  $\varphi$ , contradicting the definition of uniton number. It follows that  $\delta_1$  is full,  $\delta_i = (H_0)_{(i-1)}$  and  $\delta_i$  has rank  $i$  ( $i = 1, 2, 3$ ).

All the filtrations of Examples 3.13, 3.14 and 3.15 have length 3 and agree; further,  $\text{rank } Z_i = 4 - i$  ( $i = 0, 1, 2, 3, 4$ ), so that the legs  $\psi_i = Z_i \ominus Z_{i+1}$  are of rank one. With notation as in those examples, since  $Z_1 \neq \underline{\mathbb{C}}^4$ , either  $U_1 \neq \varphi$  or  $V_1 \neq \varphi^\perp$ ; assume without loss of generality the

former. Then we obtain the diagram (8.3), with  $Z_i = \sum_{j=i+1}^4 \psi_j$ ,  $\varphi = \psi_0 \oplus \psi_2$  and  $\varphi^\perp = \psi_1 \oplus \psi_3$ . As before the arrows show the possible non-zero second fundamental forms  $A'_{\psi_i, \psi_j}$ . The second diagram is identical to the first, but is drawn to be in keeping with our earlier diagrams, with ‘across’ arrows going down.

$$(8.3) \quad \begin{array}{ccc} \psi_0 & \longrightarrow & \psi_1 \\ \uparrow & \searrow & \uparrow \\ \psi_2 & \longrightarrow & \psi_3 \end{array} \quad \begin{array}{ccc} \psi_0 & \searrow & \psi_1 \\ \uparrow & \searrow & \uparrow \\ \psi_2 & \searrow & \psi_3 \end{array}$$

As in Example 5.4, one factorization of  $\Phi$  is provided by its Segal unitons, which we shall denote by  $\beta_i$ . According to [20, Example 4.9(i)], these are given by  $\beta_1 = h$ ,  $\beta_2 = h_{(1)}$  and

$$(8.4) \quad \beta_3 = \text{span}\{H_0 + \pi_{h_{(1)}}^\perp H_2\} \oplus G^{(1)}(h) \oplus G^{(2)}(h) = \text{span}\{H_0 + \pi_{h_{(2)}}^\perp H_2\} \oplus G^{(1)}(h) \oplus G^{(2)}(h),$$

where  $G^{(i)}(h)$  denotes the  $i$ th  $\partial'$ -Gauss transform of  $h$  (see §2.2). Hence

$$(8.5) \quad \varphi = \text{span}\{H_0 + \pi_{h_{(1)}}^\perp H_2\} \oplus G^{(2)}(h) = \text{span}\{H_0 + \pi_{h_{(2)}}^\perp H_2\} \oplus G^{(2)}(h).$$

Another factorization of  $\Phi$  is provided by the Uhlenbeck unitons, see Example 5.5; we shall denote these by  $\gamma_i$ . By the formulae in [33, Example 4.6], they are given by  $\gamma_1 = h_{(2)}$ ,  $\gamma_2 = h_{(1)}$ , and  $\gamma_3 = \text{span}\{H_0 + \pi_{h_{(2)}}^\perp H_2\}$ .

From Example 5.5, we have  $\psi_0 = \gamma_3$  and  $\psi_1 = \gamma_2^\perp \cap \gamma_1 = G^{(1)}(h)$ . It follows from (8.5) that  $\psi_2 = G^{(2)}(h)$ . From (8.4), we have  $\beta_3 = \psi_0 \oplus \psi_1 \oplus \psi_2$  so that  $\psi_3 = \beta_3^\perp$ . Thus we obtain the  $J_2$ -holomorphic twistor lift  $\psi = (\gamma_3, G^{(1)}(h), G^{(2)}(h), \beta_3^\perp) : M \rightarrow F_{1,1,1,1}$  of  $\varphi$ .

From the diagram we see that  $\varphi^\perp = \psi_1 \oplus \psi_3$  is the sum of the harmonic map  $\psi_1 = G^{(1)}(h)$  and the antiholomorphic subbundle  $\psi_3$  of  $\{\psi_1 \oplus G^{(1)}(\psi_1)\}^\perp = \psi_0 \oplus \psi_3$ , in accordance with J. Ramanathan’s description [28].

We finish with two examples: the first one real and the second one symplectic.

**Example 8.4.** Let  $H_0, H_1, H_2, H_3 : M \rightarrow \mathbb{C}^n$  be meromorphic maps, set  $\delta_1 = \text{span}\{H_0, H_2\}$ , and consider the  $\nu$ -invariant extended solution  $W = \Phi\mathcal{H}_+$  given by

$$W = \text{span}\{H_0 + \lambda^2 H_1, H_2 + \lambda^2 H_3\} + \lambda \delta_2 + \lambda^2 \delta_3 + \lambda^3 \underline{\mathcal{H}}_+,$$

where  $\underline{0} \subset \delta_1 \subset \delta_2 \subset \delta_3 \subset \underline{\mathbb{C}}^n$  is a superhorizontal sequence. By the formulae in [33, Example 4.6], we calculate the Uhlenbeck unitons as

$$(8.6) \quad \gamma_1 = \delta_3, \quad \gamma_2 = \delta_2, \quad \gamma_3 = \text{span}\{H_0 + \pi_{\delta_3}^\perp H_1, H_2 + \pi_{\delta_3}^\perp H_3\},$$

and the corresponding harmonic map  $\varphi = \Phi_{-1} : M \rightarrow G_*(\mathbb{C}^n)$  is given by  $\varphi = \gamma_3 \oplus (\delta_2^\perp \cap \delta_3)$ . From Proposition 5.2 we calculate the canonical lift  $(\psi_0, \psi_1, \psi_2, \psi_3) : M \rightarrow F$  as follows:  $\psi_0 = Z_1^\perp = \gamma_3$  and  $\psi_1 = \gamma_3^\perp \cap \gamma_2 = \delta_1^\perp \cap \delta_2$ ; since  $\varphi = \psi_0 \oplus \psi_2$ , this gives  $\psi_2 = \delta_2^\perp \cap \delta_3$ . Finally,  $\psi_3 = Z_3 = \pi_{\gamma_3}^\perp(\delta_3^\perp)$ .

Now, as in [33, §6.6], we can choose the data  $H_i$ ,  $\delta_2$ ,  $\delta_3$  such that  $W$  is real of degree 3. Then, by Proposition 6.6 we have  $\overline{\psi}_0 = \psi_3$  and  $\overline{\psi}_1 = \psi_2$ ; thus  $n$  is even, say  $n = 2m$ , and  $\varphi = \psi_0 \oplus \psi_2 = \psi_0 \oplus \overline{\psi}_1$  defines a harmonic map from  $M$  into  $\text{O}(2m)/\text{U}(m)$ . The canonical twistor lift of  $\varphi$  defined by  $\Phi$  is the  $J_2$ -holomorphic map  $(\psi_0, \psi_1, \overline{\psi}_1, \overline{\psi}_0) : M \rightarrow \text{O}(2m)/\text{U}(2) \times \text{U}(m-2)$  where  $\psi_0 = \gamma_3$  and  $\psi_1 = \gamma_3^\perp \cap \delta_2$  with  $\gamma_3$  is given by (8.6).

**Example 8.5.** Let  $H_0, H_2, H_3 : M \rightarrow \mathbb{C}^6$  be meromorphic maps with  $\text{span}\{H_0\} : M \rightarrow \mathbb{C}P^5$  full, and consider the  $\nu$ -invariant extended solution  $W = \Phi\mathcal{H}_+$  given by

$$W = \text{span}\{H_0 + \lambda^2 H_2\} + \lambda \text{span}\{(H_0)_{(1)}, H_3\} + \lambda^2 \text{span}\{(H_0)_{(2)}, (H_3)_{(1)}\} + \lambda^3 \underline{\mathcal{H}}_+.$$

Set

$$\delta_1 = \text{span}\{H_0\}, \quad \delta_2 = (H_0)_{(1)} + \text{span}\{H_3\}, \quad \delta_3 = (H_0)_{(2)} + (H_3)_{(1)}.$$

A simple calculation shows that the Uhlenbeck unitons of  $W$  are given by  $\gamma_1 = \delta_3$ ,  $\gamma_2 = \delta_2$  and  $\gamma_3 = \text{span}\{H_0 + \pi_{\delta_3}^\perp H_3\}$ . The corresponding harmonic map  $\varphi = \Phi_{-1}$  of minimal uniton number 3 is given by

$$\varphi = \gamma_3 \oplus (\delta_2^\perp \cap \delta_3) : M \rightarrow G_3(\mathbb{C}^6).$$

From Proposition 5.2, the legs of the canonical  $A_z^\varphi$ -filtration defined by  $\Phi$  are given by  $\psi_0 = \gamma_3$ ,  $\psi_1 = \delta_2 \ominus \delta_1$ ,  $\psi_2 = \delta_3 \ominus \delta_2$  and  $\psi_3 = \pi_{\gamma_3}^\perp \delta_3^\perp$ . The canonical twistor lift of  $\varphi$  defined by  $\Phi$  is the  $J_2$ -holomorphic map  $\psi = (\psi_0, \psi_1, \psi_2, \psi_3) : M \rightarrow F_{1,2,2,1}$ .

As in [33, Example 6.31] we see that  $W$  is symplectic of degree 3 if  $H_0$  is totally  $J$ -isotropic and  $H_3$  is a section of  $(H_0)_{(3)}$ . In this case,  $\varphi$  takes values in  $\text{Sp}(3)/\text{U}(3)$ , and  $J\psi_0 = \psi_3$ ,  $J\psi_1 = \psi_2$ . Then the canonical twistor lift of  $\varphi : M \rightarrow \text{Sp}(3)/\text{U}(3)$  defined by  $\Phi$  is the  $J_2$ -holomorphic map  $\psi : M \rightarrow Z_{1,2}^J = \text{Sp}(3)/\text{U}(1) \times \text{U}(2)$  given explicitly by  $\psi = (\psi_0, \psi_1, J\psi_1, J\psi_0)$  where  $\psi_0 = \text{span}\{H_0 + \pi_{\delta_3}^\perp H_3\}$  and  $\psi_1 = G'(\delta_1) + \text{span}\{H_3\}$ .

## REFERENCES

- [1] A. Bahy-El-Dien and J. C. Wood, *The explicit construction of all harmonic two-spheres in  $G_2(\mathbb{R}^n)$* , J. Reine u. Angew. Math. **398** (1989), 36–66.
- [2] A. Bahy-El-Dien and J. C. Wood, *The explicit construction of all harmonic two-spheres in quaternionic projective spaces*, Proc. London Math. Soc. (3) **62** (1991) 202–224.
- [3] J. Bolton, F. Pedit, and L. M. Woodward, *Minimal surfaces and the affine Toda field model*, J. Reine Angew. Math. **459** (1995), 119–150.
- [4] J. Bolton and L. M. Woodward, *The affine Toda equations and minimal surfaces*, in: Harmonic maps and integrable systems, 59–82, Vieweg, Braunschweig, 1994, see <http://www.amsta.leeds.ac.uk/pure/staff/wood/FordyWood/contents.html> for a downloadable version.
- [5] F. E. Burstall, *A twistor description of harmonic maps of a 2-sphere into a Grassmannian*, Math. Ann. **274** (1986), 61–74.
- [6] F. E. Burstall and M. A. Guest, *Harmonic two-spheres in compact symmetric spaces, revisited*, Math. Ann. **309** (1997), 541–572.
- [7] F. E. Burstall and J. H. Rawnsley, *Twistor theory for Riemannian symmetric spaces*, Lecture Notes in Mathematics, 1424, Springer-Verlag, Berlin, Heidelberg, 1990.
- [8] F. E. Burstall and S. M. Salamon, *Tournaments, flags, and harmonic maps*, Math. Ann. **277** (1987), 249–265.
- [9] F. E. Burstall and J. C. Wood, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom. **23** (1986), 255–298.
- [10] B. Dai and C.-L. Terng, *Bäcklund transformations, Ward solitons, and unitons*, J. Differential Geom. **75** (2007), 57–108.
- [11] J. Davidov and A. G. Sergeev, *Twistor spaces and harmonic maps* (Russian), Uspekhi Mat. Nauk **48** (1993), no. 3 (291), 3–96; translation in Russian Math. Surveys **48** (1993), no. 3, 1–91.
- [12] Y. Dong and Y. Shen, *Factorization and uniton numbers for harmonic maps into the unitary group  $\text{U}(n)$* , Sci. China Ser. A **39** (1996), 589–597.
- [13] J. Dorfmeister and J.-H. Eschenburg, *Pluriharmonic maps, loop groups and twistor theory*, Ann. Global Anal. Geom. **24** (2003), 301–321.
- [14] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.
- [15] J. Eells and S. Salamon, *Twistorial construction of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), 589–640 (1986).
- [16] J. Eells and J. C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Advances in Math. **49** (1983), 217–263.
- [17] S. Erdem and J. C. Wood, *On the constructions of harmonic maps into a Grassmannian*, J. London Math. Soc. (2), **28** (1983), 161–174.
- [18] J.-H. Eschenburg and R. Tribuzy, *Associated families of pluriharmonic maps and isotropy*, Manuscripta Math. **95** (1998), 295–310.
- [19] M. J. Ferreira and B. A. Simões, *Explicit construction of harmonic two-spheres into the complex Grassmannian*, preprint, arXiv:1007.4143 (2010).
- [20] M. J. Ferreira, B. A. Simões and J. C. Wood, *All harmonic 2-spheres in the unitary group, completely explicitly*, Math. Z. **266** (2010), 953–978.
- [21] M. A. Guest, *Harmonic maps, loop groups, and integrable systems*, London Mathematical Society Student Texts, 38, Cambridge University Press, Cambridge, 1997.
- [22] M. A. Guest, *An update on harmonic maps of finite uniton number, via the zero curvature equation*, Integrable systems, topology, and physics (Tokyo, 2000), 85–113, Contemp. Math. **309**, Amer. Math. Soc., Providence, RI, 2002.
- [23] Q. He and Y. B. Shen, *Explicit construction for harmonic surfaces in  $\text{U}(N)$  via adding unitons*, Chinese Ann. Math. Ser. B **25** (2004), 119–128.
- [24] J. L. Koszul and B. Malgrange, *Sur certaines structures fibrées complexes*, Arch. Math. **9** (1958), 102–109.

- [25] R. Pacheco, *Harmonic two-spheres in the symplectic group  $\mathrm{Sp}(n)$* , Internat. J. Math. **17** (2006), 295–311.
- [26] R. Pacheco, *On harmonic tori in compact rank one symmetric spaces*, Diff. Geom. Appl. **27** (2009), 352–361.
- [27] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 1986.
- [28] J. Ramanathan, *Harmonic maps from  $S^2$  to  $G_{2,4}$* , J. Differential Geom. **19** (1984), 207–219.
- [29] J. Rawnsley, *F-structures, f-twistor spaces and harmonic maps*, Geometry seminar “Luigi Bianchi” II–1984, 85–159, Lecture Notes in Math., 1164, Springer, Berlin, 1985.
- [30] S. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.
- [31] S. Salamon, *Harmonic and holomorphic maps*, Geometry Seminar “Luigi Bianchi” II–1984, 161–224, Lecture Notes in Math., 1164, Springer, Berlin (1985).
- [32] G. Segal, *Loop groups and harmonic maps*, Advances in homotopy theory (Cortona, 1988), 153–164, London Math. Soc. Lecture Notes Ser., 139, Cambridge Univ. Press, Cambridge, 1989.
- [33] M. Svensson and J. C. Wood, *Filtrations, factorizations and explicit formulae for harmonic maps*, Comm. Math. Phys. (to appear).
- [34] K. Uhlenbeck, *Harmonic maps into Lie groups: classical solutions of the chiral model*, J. Differential Geom. **30** (1989), 1–50.
- [35] J. G. Wolfson, *Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds*, J. Differential Geom. **27** (1988), 161–178.
- [36] J. C. Wood, *Explicit construction and parametrization of harmonic two-spheres in the unitary group*, Proc. London Math. Soc. (3) **58** (1989), 608–624.
- [37] J. C. Wood, *Explicit constructions of harmonic maps*, in: Harmonic Maps and Differential Geometry, ed. E. Loubeau and S. Montaldo, Contemp. Math., 542, Amer. Math. Soc. (2011), 41–74.

DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE, UNIVERSITY OF SOUTHERN DENMARK, AND CP3–ORIGINS, CENTRE OF EXCELLENCE FOR PARTICLE PHYSICS AND PHENOMENOLOGY, CAMPUSVEJ 55, DK-5230 ODENSE M, DENMARK

*E-mail address:* `svensson@imada.sdu.dk`

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF LEEDS, LEEDS LS2 9JT, GREAT BRITAIN

*E-mail address:* `j.c.wood@leeds.ac.uk`